Two Interacting Electrons in a Disorder Potential: Localization Properties

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Abstract

Two electrons move in a quasi one–dimensional wire under the influence of a short–range interaction. We restrict Hilbert space to those states where the two electrons are close to each other. Using supersymmetry, we present a complete analytical solution to this problem. The two–body interaction affects the density of states and, thereby, the localization length. We derive a criterion for the onset of changes of the localization length due to the two–body interaction.

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1 Introduction

The eigenfunctions of electrons moving in a random disorder potential in one or two dimensions display localization. The influence of the Coulomb interaction (or of any other two–body interaction) on localization properties has been an open problem for a long time. In 1994, Shepelyansky [1] investigated the motion of two interacting electrons in a disordered one–dimensional wire numerically. He found that the two–body interaction may strongly increase the localization length, independently of the sign of the interaction. This surprising result was rederived by Imry [2] who used Thouless' block scaling picture. Further numerical work [3, 4, 5, 6, 7, 8, 9] in one dimension aimed at a better understanding and quantification of these findings, and at settling a controversy which had arisen. As a result, the effect is now well established. In addition, Song and von Oppen [9] have demonstrated that it depends strongly on the distance between the two electrons and is strongest when these remain close together.

It is obviously desirable to extend these investigations to two–dimensional systems. At present, numerical approaches do not seem capable of handling the ensuing difficulties. However, the work of Refs. [1, 2, 8] yields an analytical expression for the change of the localization length $\zeta(U)$ under the influence of a two–body interaction with typical matrix element U. The result

$$\frac{\zeta(U)}{\zeta(0)} - 1 \propto \left(\frac{U}{B}\right)^2 , \tag{1}$$

with B the bandwidth due to the disorder potential, is independent of the sign of the two–body interaction and is claimed to hold both in one and in two dimensions. Numerical results in one dimension [6], although in line with the sign–independence, do not fully support the U^2 dependence predicted analytically. Moreover, the effect is numerically found to be strongest when the two electrons move at short distance [9], a feature which is also not covered by the analytical expression. Finally, the result (1) has either been derived heuristically [2], or it is based on statistical assumptions [8] which, although plausible, are not obviously valid.

This situation calls for a novel approach to the problem. In the present paper, we extend the study of localization properties of two interacting electrons to the case where the electrons move in a quasi one–dimensional disordered wire. This case is obviously more realistic (and numerically much more difficult) than that of the strictly one–dimensional problem. Therefore, we aim at an analytical treatment. The electrons interact via a short–range interaction. The realistic case of the Coulomb interaction is supposed to be included in our treatment because screening removes the long–range part of that interaction. We use an analytical method developed recently for the study of the k–body embedded ensembles of random matrices [11]. Motivated by the numerical results of Ref. [9], we reduce Hilbert space and keep only states where the two electrons stay reasonably close together. This leads to a non–linear sigma model. We determine the saddle–point manifold and thereby the influence of the two–body interaction on localization. We investigate all terms in the loop expansion up to second

order in the hopping matrix elements and show that these provide the expected renormalization of the saddle–point values. We derive a criterion for the onset of interaction–induced changes in the localization length.

A brief account of this work omitting most technical details was published in Ref. [10].

2   Model

We treat the quasi one–dimensional wire in the manner introduced in Ref. [12]. The wire of length L is considered as being divided into K slices labelled $a, b, c, \ldots, K$. The surface connecting neighboring slices is transverse to the direction of the current through the wire. We eventually consider the limit in which the longitudinal length of the slices tends to zero and, simultaneously, K tends to infinity. In each slice the actual disorder is replaced by a random single–particle Hamiltonian. To simplify the notation and algebra, we consider the case of unitary symmetry. Hopping matrix elements connect neighboring slices and allow an electron to move from slice a to slices $a \pm 1$. This model has been successfully used to calculate universal conductance fluctuations [12] and, later, the localization properties of both, the average conductance and the variance of the conductance [13, 14]. The interaction between the two electrons moving in the disordered wire vanishes unless both electrons occupy the same slice.

In slice a we take an arbitrary basis of single–particle states labelled $|aj\rangle$, with $j = 1, 2, \ldots, l$. We later take the limit $l \to \infty$. The associated creation and annihilation operators are denoted by $\alpha_{aj}^\dagger$ and α_{aj}, respectively. The Hilbert space of the two–electron problem is spanned by the orthonormal states $|aibj\rangle$ defined by

$$|aibj\rangle = \alpha_{ai}^\dagger \alpha_{bj}^\dagger |0\rangle , \qquad (2)$$

where $|0\rangle$ denotes the vacuum. This definition is unique if we require $a \leq b$ and, for $a = b$, $i < j$. For $a < b$ fixed, the number N_{ab} of states is l^2 while for $a = b$, we have $N_{aa} = l(l-1)/2$. We always take $l \gg 1$ and use $N_{aa} = l^2/2$.

The Hamiltonian H is the sum of three terms,

$$H = H_0 + H_1 + H_2 . \qquad (3)$$

The single–particle Hamiltonian

$$H_0 = \sum_{aij} h_{ij}^{(a)} \alpha_{ai}^\dagger \alpha_{aj} \qquad (4)$$

describes the motion of the electron within each slice under the influence of the disorder potential. The Hermitean matrices $h^{(a)}$ are members of the Gaussian unitary ensemble of random matrices. They have dimension l, mean value zero and second moments given by

$$\overline{h_{ij}^{(a)} h_{i'j'}^{(a')*}} = \frac{\lambda^2}{l} \delta_{aa'} \delta_{ii'} \delta_{jj'} . \qquad (5)$$

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The overbar denotes the ensemble average. The parameter λ has the dimension of an energy and is independent of the slice label a. The single–particle spectrum in each slice has the shape of a semicircle with radius 2λ. Hopping between neighboring slices is described by the Hamiltonian

$$H_1 = \sum_{aij} [v_{ij}^{(a)} \alpha_{ai}^{\dagger} \alpha_{a+1j} + h.c.] \ . \tag{6}$$

It was shown in Ref. [12] that the hopping matrix elements $v^{(a)}$ may either be chosen as Gaussian–distributed complex random variables with mean value zero and a second moment $\overline{v_{ij}^{(a)} v_{i'j'}^{(a')*}} = \frac{v^2}{l} \delta_{aa'} \delta_{ii'} \delta_{jj'}$ or, equivalently, as elements of a constant diagonal matrix,

$$v_{ij}^{(a)} = v \ \delta_{ij} \ . \tag{7}$$

Here, v is real and independent of the slice label a. In the present context, we adopt the second alternative. The two–body interaction H_2 is fixed (not random) and has the same matrix elements within each slice,

$$H_2 = \sum_{a} \sum_{i<j, i'<j'} w_{iji'j'} \alpha_{ai}^{\dagger} \alpha_{aj}^{\dagger} \alpha_{aj'} \alpha_{ai'} \ . \tag{8}$$

The two–body matrix elements are antisymmetric in the pairs (ij) and $(i'j')$ and Hermitean.

We wish to determine how the two–body interaction (8) affects localization properties. The effect (if any) will be most dramatic if both electrons stay close to each other as they traverse the wire. In order to focus on this point, and in order to simplify the calculation, we reduce the Hilbert space of two–particle states. We admit only states where both electrons are in the same slice, or in neighboring slices. Thus, of all the states introduced in Eq. (2), we keep only the ones with $b = a$ and with $b = (a + 1)$. Without this omission, our model would be very general. It is, therefore, important to address the physical significance of this simplification. The numerical work of Ref. [9] in one dimension has clearly demonstrated that the change of the localization length caused by the two–body interaction is biggest when the two electrons which move through the one–dimensional wire, keep the shortest distance from each other. This is our motivation for the constraint artificially imposed in our model. It might have been desirable to loosen the constraint and to allow the electrons to keep a maximum distance given by the localization length in the absence of disorder. However, this was technically impossible. Therefore, we cannot claim that our results are quantitatively correct. However, in view of the work of Ref. [9], we believe that we obtain qualitatively correct answers. The shortcoming of our approach – the reduction of Hilbert space – must be weighed against the fact that with this simplification, we are able to obtain a complete analytical solution of and a physically transparent answer to the problem.

We aim at a comparison of the localization properties of this simplified system when the two–body interaction (8) is either turned off or fully present.

Localization properties can be read off the two–point correlation function

$$C(n) = \overline{|\langle a(a+1)\mu_0|(E^+ - H)^{-1}|(a+n)(a+n+1)\nu_0\rangle|^2} \ . \qquad (9)$$

The indices μ_0 and ν_0 are defined in Eq. (10) below. To be independent of edge effects, we choose $1 \ll a \ll (a+n) \ll K$. For large n, $C(n)$ should decay exponentially with n. In proper units, the coefficient in the exponent defines the localization length. We calculate $C(n)$ using supersymmetry [15, 16].

3 Second Moment of the One–Body Hamiltonian H_0. Eigenvalue Expansion

The matrix elements of the single–particle Hamiltonian H_0 in the Hilbert space of two–electron states (2) with $b = a$ and $b = (a+1)$ are Gaussian–distributed random variables with mean value zero. Therefore, the distribution of these matrix elements is fully determined by their second moments. We follow Refs. [11] and consider these second moments as matrices in the product space of two Hilbert–space vectors. We use this representation to derive the eigenvalue expansion of the second moments. This will allow us to apply the supersymmetry technique.

The operator H_0 does not change the number of particles per slice. Therefore, the operator $\alpha_{ak}^\dagger \alpha_{ak'}$ occurs in the following three types of matrix elements: $\langle (a-1)iaj|H_0|(a-1)i'aj'\rangle$, $\langle aiaj|H_0|ai'aj'\rangle$, and $\langle ai(a+1)j|H_0|ai'(a+1)j'\rangle$, with $a = 1, \ldots, K$. Here and in the sequel, we write the adjoint of the Hilbert vector $|aibj\rangle$ as $\langle aibj|$. We simplify the notation by grouping the state labels into pairs,

$$\mu = (i_1 j_1), \ \sigma = (i_1' j_1'), \ \rho = (i_2 j_2), \ \nu = (i_2' j_2') \ . \qquad (10)$$

Thus, the Hilbert vector $|aibj\rangle$ is written as $|ab\mu\rangle$. In the sequel, we refer to the set of states $|ab\mu\rangle$ encompassing all values of μ as to a box.

There are nine types of terms contributing to the second moment. Three of them have "diagonal form" and are given by

$$\frac{l}{\lambda^2} \overline{\langle (a-1)a\mu|H_0|(a-1)a\sigma\rangle\langle (a-1)a\rho|H_0|(a-1)a\nu\rangle}$$

$$= \mathcal{A}_{\mu\nu;\rho\sigma}^{(-1)} = \sum_{ckk'}\langle (a-1)a\mu|\alpha_{ck}^\dagger \alpha_{ck'}|(a-1)a\sigma\rangle$$

$$\times \langle (a-1)a\rho|\alpha_{ck'}^\dagger \alpha_{ck}|(a-1)a\nu\rangle \ , \qquad (11)$$

$$\frac{l}{\lambda^2} \overline{\langle aa\mu|H_0|aa\sigma\rangle\langle aa\rho|H_0|aa\nu\rangle} = \mathcal{A}_{\mu\nu;\rho\sigma}^{(0)}$$

$$= \sum_{kk'}\langle aa\mu|\alpha_{ak}^\dagger \alpha_{ak'}|aa\sigma\rangle\langle aa\rho|\alpha_{ak'}^\dagger \alpha_{ak}|aa\nu\rangle \ , \qquad (12)$$

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$$\frac{l}{\lambda^2} \overline{\langle a(a+1)\mu|H_0|a(a+1)\sigma\rangle\langle a(a+1)\rho|H_0|a(a+1)\nu\rangle}$$

$$= \mathcal{A}_{\mu\nu;\rho\sigma}^{(+1)} = \sum_{ckk'}\langle a(a+1)\mu|\alpha_{ck}^\dagger\alpha_{ck'}|a(a+1)\sigma\rangle$$

$$\times\langle a(a+1)\rho|\alpha_{ck'}^\dagger\alpha_{ck}|a(a+1)\nu\rangle . \tag{13}$$

The sum over c in Eqs. (11,13) runs over $((a-1),a)$ and $(a,(a+1))$, respectively. We observe that $\mathcal{A}_{\mu\nu;\rho\sigma}^{(+1)}$ is obtained from $\mathcal{A}_{\mu\nu;\rho\sigma}^{(-1)}$ by shifting $a \to (a+1)$. Therefore, we consider only $\mathcal{A}_{\mu\nu;\rho\sigma}^{(-1)}$ and $\mathcal{A}_{\mu\nu;\rho\sigma}^{(0)}$ in the sequel. The "non–diagonal" contributions to the second moment are

$$\frac{l}{\lambda^2} \overline{\langle (a-1)a\mu|H_0|(a-1)a\sigma\rangle\langle aa\rho|H_0|aa\nu\rangle}$$

$$= \mathcal{A}_{\mu\nu;\rho\sigma}^{(-1,0)} = \sum_{kk'}\langle (a-1)a\mu|\alpha_{ak}^\dagger\alpha_{ak'}|(a-1)a\sigma\rangle$$

$$\times\langle aa\rho|\alpha_{ak'}^\dagger\alpha_{ak}|aa\nu\rangle , \tag{14}$$

$$\frac{l}{\lambda^2} \overline{\langle (a-1)a\mu|H_0|(a-1)a\sigma\rangle\langle a(a+1)\rho|H_0|a(a+1)\nu\rangle}$$

$$= \mathcal{A}_{\mu\nu;\rho\sigma}^{(-1,+1)} = \sum_{kk'}\langle (a-1)a\mu|\alpha_{ak}^\dagger\alpha_{ak'}|(a-1)a\sigma\rangle$$

$$\times\langle a(a+1)\rho|\alpha_{ak'}^\dagger\alpha_{ak}|a(a+1)\nu\rangle , \tag{15}$$

$$\frac{l}{\lambda^2} \overline{\langle aa\mu|H_0|aa\sigma\rangle\langle a(a+1)\rho|H_0|a(a+1)\nu\rangle}$$

$$= \mathcal{A}_{\mu\nu;\rho\sigma}^{(0,+1)} = \sum_{kk'}\langle aa\mu|\alpha_{ak}^\dagger\alpha_{ak'}|aa\sigma\rangle$$

$$\times\langle a(a+1)\rho|\alpha_{ak'}^\dagger\alpha_{ak}|a(a+1)\nu\rangle , \tag{16}$$

and the three Hermitean conjugate forms.

Eqs. (14) to (16) show that the matrix elements of H_0 taken in different boxes, are correlated. This is a trivial consequence of the fact that one of the two electrons occupies the same slice. The correlation causes our problem to differ from that of the motion of a single elctron through the wire. We return to this point in Sections 9 and 10 below.

In order to be able to apply the supersymmetry technique, we now construct the eigenvalue expansions of the matrices $\mathcal{A}$ introduced in Eqs. (11) to (16). We essentially follow Ref. [11]. We explicitly consider $\mathcal{A}_{\mu\nu;\rho\sigma}^{(-1)}$ and $\mathcal{A}_{\mu\nu;\rho\sigma}^{(-1,+1)}$ and only state the results for the remaining matrices $\mathcal{A}$.

The matrix $\mathcal{A}_{\mu\nu;\rho\sigma}^{(-1)}$ is Hermitean,

$$(\mathcal{A}_{\sigma\rho;\nu\mu}^{(-1)})^* = \mathcal{A}_{\mu\nu;\rho\sigma}^{(-1)} . \tag{17}$$

Therefore, $\mathcal{A}^{(-1)}$ can be expanded in terms of its eigenvalues and eigenvectors. The eigenvalue equation reads

$$\sum_{\rho\sigma} \mathcal{A}^{(-1)}_{\mu\nu;\rho\sigma} C^{(s\tau)}_{\sigma\rho}(-1) = \Lambda^{(s)}(-1) C^{(s\tau)}_{\mu\nu}(-1) . \tag{18}$$

Here s labels the different eigenvalues, and τ distinguishes degenerate eigenvectors. The latter are normalized according to

$$\sum_{\mu\nu} C^{(s\tau)}_{\mu\nu}(-1)(C^{(s'\tau')}_{\mu\nu}(-1))^\dagger = N_{(a-1)a} \, \delta_{ss'}\delta_{\tau\tau'} \tag{19}$$

where $N_{(a-1)a}$ labels the dimension of the matrix space, see Section 2. If the eigenvectors form a complete set, the eigenvalue expansion of $\mathcal{A}^{(-1)}$ is given by

$$\mathcal{A}^{(-1)}_{\mu\nu;\rho\sigma} = \frac{1}{N_{(a-1)a}} \sum_{s\tau} \Lambda^{(s)} C^{(s\tau)}_{\mu\nu}(-1)(C^{(s\tau)}_{\sigma\rho}(-1))^\dagger . \tag{20}$$

According to Ref. [11] we expect that the sum over s extends over $s = 0, 1$ only. This is because H_0 is a one–body operator and we deal with two electrons. (It was shown in Ref. [11] that for a general k–body operator and m Fermions, the sum over s runs from 0 to $(m - k)$). Moreover, the eigenvectors to $s = 0$ are expected not to be degenerate and generically to have the form $C^{(0)}_{\mu\nu} \propto \delta_{\mu\nu} = \delta_{i_1 i_2'}\delta_{j_1 j_2'}$ while the eigenvectors to $s = 1$ are generically expected to be $(l^2 - 1)$– fold degenerate and to be given by $C^{(1\tau)}_{\mu\nu}(-1) \propto \langle(a-1)a\mu|\alpha^\dagger_{cm}\alpha_{cm'}|(a-1)a\nu\rangle$, with $c = (a - 1), a$. The label τ stands for all combinations of indices (m, m') with $m \neq m'$. For $m = m'$, we have $C^{(1\tau)}_{\mu\nu}(-1) \propto \langle(a-1)a\mu|(\alpha^\dagger_{cm}\alpha_{cm} - (1/l)\sum^l_{n=1}\alpha^\dagger_{cm}\alpha_{cm})|(a-1)a\nu\rangle$ with $c = (a-1), a$. Using this ansatz, we do indeed solve the eigenvalue equation. The eigenvalues are

$$\Lambda^{(0)}(-1) = 2l \; ; \; \Lambda^{(1)}(-1) = (2l - 1) . \tag{21}$$

Eigenvalues with $s > 1$ vanish. Counting shows that the eigenvectors form a complete set. Thus, the expansion (20) is established.

We turn to $\mathcal{A}^{(-1,0)}_{\mu\nu;\rho\sigma}$ in Eq. (14). This matrix does not have the property (17). Nevertheless, an expansion analogous to Eq. (20) can be found. We first determine the right–hand and left–hand eigenvectors. It is obvious that eigenvectors to $s = 0$ do not exist. The right–hand eigenvectors $C^{(1\tau)}(-1, 0; r)$ to $s = 1$ are proportional to $\langle(a-1)a\mu|\alpha^\dagger_{(a-1)m}\alpha_{am'}|aa\nu\rangle$ and belong to eigenvalue

$$\Lambda^{(1)}(-1, 0) = l - 1 . \tag{22}$$

The left–hand eigenvectors $C^{(1\tau)}(-1, 0; l)$ to the same eigenvalue are proportional to $\langle aa\mu|\alpha^\dagger_{am}\alpha_{(a-1)m'}|(a-1)a\nu\rangle$. All eigenvalues with $s > 1$ vanish. Each pair of left– and right–hand eigenvectors fulfills orthogonality relations analogous to Eq. (19) and can be normalized correspondingly. As a result, $\mathcal{A}^{(-1,0)}$ can be written in the form

$$\mathcal{A}^{(-1,0)}_{\mu\nu;\rho\sigma} = \frac{1}{N(-1, 0)}(l - 1) \sum_\tau C^{(1\tau)}_{\mu\nu}(-1, 0; l)C^{(1\tau)}_{\sigma\rho}(-1, 0; r) . \tag{23}$$

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The factor N is the normalization factor. For the remaining five matrices $\mathcal{A}$, we proceed analogously. The eigenvectors and eigenvalues are listed in Appendix 1. In summary, all matrices $\mathcal{A}^{(i)}$ with

$$i = -1, 0, +1, (-1, 0), (-1, +1), (0, +1), (+1, 0), (+1, -1), (0, -1) \qquad (24)$$

possess the expansion

$$\mathcal{A}^{(i)}_{\mu\nu;\rho\sigma} = \frac{1}{N(i)} \sum_{s=0}^{1} \Lambda^{(s)}(i) \sum_{\tau} C^{(s\tau)}_{\mu\nu}(i; l) C^{(s\tau)}_{\rho\sigma}(i; r)) . \qquad (25)$$

We observe that the left–and right–hand eigenvectors are Hermitean adjoints of each other. In the sequel, we will need the explicit values of the normalization factors $N(0) = l^2/2$ and $N(1) = l^2$.

4   Supersymmetry. Generating Function

A discussion of the localization properties of the two–electron system requires the knowledge of the average two–point function. We calculate this function using Efetov's supersymmetry technique [15] and the notation of Ref. [16]. The technique has found wide applications in the theory of disordered solids. Therefore, we do not give many details. We confine ourselves to indicating where the present treatment differs from that of Refs. [11].

The generating functional Z is an integral over commuting and anticommuting variables. These are arranged in the form of the graded vectors $\Psi_{ab\mu;\alpha}$ which carry the labels (ab) of the boxes, the running label μ within each box, and the label α which for every value of $(ab\mu)$ distinguishes two complex commuting ($\alpha = 1, 3$) and two anticommuting ($\alpha = 2, 4$) integration variables. The functional Z contains in the exponent of the integrand a term arising from H_0. This term has the form

$$-\frac{i}{2} \sum_{ab} \sum_{\mu\nu} \Psi^*_{ab\mu} L^{1/2} \langle ab\mu | H_0 | ab\nu \rangle L^{1/2} \Psi_{ab\nu} . \qquad (26)$$

The diagonal graded matrix L is given by $\mathrm{diag}(1, 1, -1, -1)$. The graded indices have been omitted in Eq. (26). After averaging over the ensemble $\{H_0\}$, the generating functional contains in the exponent a term given by $1/2$ times the mean value of the square of the expression (26). This term is worked out in Appendix 2. As a result, we find

$$\frac{1}{2} \overline{\left(-\frac{i}{2} \sum_{ab} \sum_{\mu\nu} \Psi^*_{ab\mu} L^{1/2} \langle ab\mu | H_0 | ab\nu \rangle L^{1/2} \Psi_{ab\nu} \right)^2}$$

$$= \frac{1}{2l^2} \sum_{a=1}^{K} \mathrm{trg}[(A^{(aa)}(0))^2] + \frac{1}{4l^2} \sum_{a=1}^{K-1} \mathrm{trg}[(A^{(a(a+1))}(0))^2]$$

$$
+ \quad \frac{1}{4l^2} \sum_{a=1}^{K} \sum_{\tau} \mathrm{trg}[(A^{(aa;\tau)}(1))^2] + \frac{1}{4l^2} \sum_{a=1}^{K-1} \sum_{\tau} \mathrm{trg}[(A^{(a(a+1);\tau)}(1))^2]
$$

$$
+ \quad \frac{1}{4} \sum_{a=2}^{K} N(-1,0)^{-1} \sum_{\tau} \mathrm{trg}[A^{((a-1)a;\tau)}(-1,0;l)A^{((a-1)a;\tau)}(-1,0;r)]
$$

$$
+ \quad \frac{1}{4} \sum_{a=2}^{K-1} N(-1,1)^{-1} \sum_{\tau} \mathrm{trg}[A^{((a-1)(a+1);\tau)}(-1,1;l)A^{((a-1)(a+1);\tau)}(-1,1;r)]
$$

$$
+ \quad \frac{1}{4} \sum_{a=1}^{K-1} N(1,0)^{-1} \sum_{\tau} \mathrm{trg}[A^{(a(a+1);\tau)}(0,1;l)A^{(a(a+1);\tau)}(0,1;r)] \ . \tag{27}
$$

The symbols A appearing on the right–hand side denote graded matrices which are defined in Appendix 2. Each of the matrices A is bilinear in the integration variables Ψ. As a consequence, the expression (27) is quartic in the Ψ's. To obtain bilinear expressions and work out the Ψ–integrals, one usually performs a Hubbard–Stratonovich (HS) transformation. Here, this step is slightly more complicated than is the case usually because some of the terms in expression (27) contain bilinear expressions in (rather than squares of) the matrices A. To overcome this problem, we recall that – with proper identification of the indices τ – the matrices $C_{\mu\nu}^{1\tau}(-1,0;r)$ and $C_{\mu\nu}^{1\tau}(-1,0;l)$ are Hermitean adjoints of each other, see Eqs. (101). Therefore, the matrices

$$
\begin{aligned}
D^{((a-1)a;\tau)} &= \frac{1}{2}[A^{((a-1)a;\tau)}(-1,0;r) - A^{((a-1)a;\tau)}(-1,0;l)] \ , \\
E^{((a-1)a;\tau)} &= \frac{1}{2i}[A^{((a-1)a;\tau)}(-1,0;r) + A^{((a-1)a;\tau)}(-1,0;l)]
\end{aligned} \tag{28}
$$

have the same symmetry properties as the matrices $A^{(aa)}(0)$. On the other hand, using the cyclic invariance of the trace, we have

$$
\begin{aligned}
&\mathrm{trg}[A^{((a-1)a;\tau)}(-1,0;l)A^{((a-1)a;\tau)}(-1,0;r)] \\
&= \mathrm{trg}[(iD^{((a-1)a;\tau)})^2 + (iE^{((a-1)a;\tau)})^2] \ .
\end{aligned} \tag{29}
$$

Therefore, the usual HS transformation can now be used for D and E. We proceed correspondingly for $A^{((a-1)(a+1);\tau)}(-1,1;r)$ and $A^{(a(a+1);\tau)}(0,1;r)$. For every possible combination of slice labels (a,b) and for each of the graded matrices A we introduce a corresponding graded 4×4 σ–matrix with the following correspondence:

$$
\begin{aligned}
A^{(ab)} &\leftrightarrow \sigma^{(ab)} \ , \\
A^{(ab;\tau)} &\leftrightarrow \sigma^{(ab;\tau)} \ , \\
D^{(ab;\tau)} &\leftrightarrow \sigma^{(ab;\tau)}(1) \ , \\
E^{(ab;\tau)} &\leftrightarrow \sigma^{(ab;\tau)}(2) \ .
\end{aligned} \tag{30}
$$

The volume element for integration over all these variables is denoted by $\mathrm{d}[\sigma]$.

As a result, the ensemble average of the generating function Z takes the form

$$\overline{Z} = \int d[\sigma] \exp\left[-\sum_a \text{trg}\left(\frac{l^2}{8} (\sigma^{(aa)})^2 + \frac{l^2}{4} (\sigma^{(a((a+1)))})^2 \right.\right.$$

$$+ \sum_\tau \left\{ \frac{l^2}{4} (\sigma^{(aa;\tau)})^2 + \frac{l^2}{4} (\sigma^{(a(a+1);\tau)})^2 \right.$$

$$+ \sum_{i=1}^{2} \left(\frac{N(-1,0)}{4} (\sigma^{((a-1)a;\tau)}(i))^2 + \frac{N(-1,1)}{4} (\sigma^{((a-1)(a+1);\tau)}(i))^2 \right.$$

$$\left.\left.\left. + \frac{N(0,1)}{4} (\sigma^{(a(a+1);\tau)}(i))^2 \right\} \right) - \text{trg tr ln } \mathbf{N}(J) \right] . \tag{31}$$

Here $\mathbf{N}(J)$ is both, a graded matrix and a matrix in Hilbert space with basis vectors $|aa\mu\rangle$ and $|a(a+1)\nu\rangle$. In order to clearly display the structure of $\mathbf{N}(J)$, we list separately several types of matrix elements. There are two types of diagonal matrix elements, the first type given by

$$\langle aa\mu|\mathbf{N}(J)| \; aa\nu\rangle = \left(E - \lambda\sigma^{(aa)} \right)\delta_{\mu\nu}$$

$$- \sum_\tau \lambda\sigma^{(aa;\tau)}C_{\mu\nu}^{(1\tau)}(0) \; - w_{\mu\nu} \; . \tag{32}$$

As usual, we denote by E half the sum of the energy arguments of both Green's functions. The difference vanishes as both energies are equal. The matrix J denotes the source matrix. The second type of diagonal matrix element is

$$\langle a(a+1)\mu|\mathbf{N}(J)| \; a(a+1)\nu\rangle = \left(E - \lambda\sigma^{(a(a+1))} \right) \delta_{\mu\nu}$$

$$- \sum_\tau \lambda\sigma^{(a(a+1);\tau)}C_{\mu\nu}^{(1\tau)}(1) \; . \tag{33}$$

The non–diagonal matrix elements of $\mathbf{N}(J)$ connect either neighboring two–particle states,

$$\langle (a-1)a\mu|\mathbf{N}(J)|aa\nu\rangle = -v \; \delta_{\mu\nu}$$

$$- (1/2)\sum_\tau \lambda C_{\mu\nu}^{(1\tau)}(-1,0;l)(-i\sigma^{((a-1)a;\tau)}(1) + \sigma^{((a-1)a;\tau)}(2)) \; , \tag{34}$$

$$\langle aa\mu|\mathbf{N}(J)|a(a+1)\nu\rangle = -v \; \delta_{\mu\nu}$$

$$- (1/2)\sum_\tau \lambda C_{\mu\nu}^{(1\tau)}(0,1;l)(-i\sigma^{(a(a+1);\tau)}(1) + \sigma^{(a(a+1);\tau)}(2)) \; , \tag{35}$$

$$\langle aa\mu|\mathbf{N}(J)|(a-1)a\nu\rangle = -v \; \delta_{\mu\nu}$$

$$- (1/2)\sum_\tau \lambda C_{\mu\nu}^{(1\tau)}(-1,0;r)(i\sigma^{((a-1)a;\tau)}(1) + \sigma^{((a-1)a;\tau)}(2)) \; , \tag{36}$$

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$$\langle a(a+1)\mu|\mathbf{N}(J)|aa\nu\rangle = -v\ \delta_{\mu\nu}$$
$$-(1/2)\sum_{\tau}\lambda C_{\mu\nu}^{(1\tau)}(0,1;r)(i\sigma^{(a(a+1);\tau)}(1)+\sigma^{(a(a+1);\tau)}(2))\ ,\quad (37)$$

or next–nearest states,

$$\langle(a-1)a\mu|\mathbf{N}(J)|a(a+1)\nu\rangle$$
$$=-(1/2)\sum_{\tau}\lambda C_{\mu\nu}^{(1\tau)}(-1,1;l)\left(-i\sigma^{((a-1)(a+1);\tau)}(1)\right.$$
$$\left.+\sigma^{((a-1)(a+1);\tau)}(2)\right)\ ,\quad (38)$$

$$\langle(a+1)a\mu|\mathbf{N}(J)|a(a-1)\nu\rangle$$
$$=-(1/2)\sum_{\tau}\lambda C_{\mu\nu}^{(1\tau)}(-1,1;r)\left(i\sigma^{((a-1)(a+1);\tau)}(1)\right.$$
$$\left.+\sigma^{((a-1)(a+1);\tau)}(2)\right)\ .\quad (39)$$

The matrix $\mathbf{N}(J)$ also contains the source terms J. These depend upon the observable we are interested in. For the case of localization properties of the two–point function, we define the source terms in Section 7. All other matrix elements of $\mathbf{N}(J)$ vanish.

5 Saddle–Point Equations

For an approximate evaluation of the averaged generating function $\overline{Z}$, we follow standard procedure and use the saddle–point approximation. This is suggested by the occurrence of large factors l^2 and $N(i)$ in front of the quadratic terms in the exponent of Eq. (31). We neglect J which is infinitesimal. Following the work of Ref. [12], we also neglect the hopping terms v. For the single–electron problem with disorder, it was shown in Ref. [12] that by treating v as a small perturbation of the saddle–point solution, the standard non–linear sigma model for localization is obtained. Technically speaking we assume that $v \ll \lambda$. The matrix obtained from $\mathbf{N}(J)$ after all these omissions is denoted by $\mathbf{N}'(0)$.

In the exponent of Eq. (31), we replace every σ–matrix by $\sigma+\delta\sigma$. We collect the terms linear in $\delta\sigma$ and put the coefficients multiplying the independent $\delta\sigma$'s equal to zero. As a result, we find

$$\sigma^{(aa)} = \frac{4}{l^2}\ \mathrm{tr}\left[\left(\frac{1}{\mathbf{N}'(0)}\right)_{(aa|aa)}\lambda\right]\ .\quad (40)$$

The index $(aa|aa)$ indicates that the diagonal aa box of the inverse of $\mathbf{N}'(0)$ has to be taken. The trace is taken with respect to the state indices μ,ν in this box. Quite generally, we use the expression "box" to denote the totality of Hilbert

states obtained by putting the two electrons in slices a and b, respectively, with $b = a$ or $b = (a+1)$. Using analogous notation, we find

$$\sigma^{(aa;\tau)} = \frac{2}{l^2} \, \text{tr}\left[\left(\frac{1}{\mathbf{N}'(0)}\right)_{(aa|aa)} \lambda C^{(1\tau)}(0)\right],$$

$$\sigma^{(a(a+1))} = \frac{2}{l^2} \, \text{tr}\left[\left(\frac{1}{\mathbf{N}'(0)}\right)_{(a(a+1)|a(a+1))} \lambda\right],$$

$$\sigma^{(a(a+1);\tau)} = \frac{2}{l^2} \, \text{tr}\left[\left(\frac{1}{\mathbf{N}'(0)}\right)_{(a(a+1)|a(a+1))} \lambda C^{(1\tau)}(1)\right]. \qquad (41)$$

For the terms which are not diagonal in the box indices, it is convenient to introduce the matrices

$$\Sigma^{(ab;\tau)} = -\sigma^{(ab;\tau)}(1) - i\sigma^{(ab;\tau)}(2),$$

$$(\Sigma^{(ab;\tau)})^\dagger = -\sigma^{(ab;\tau)}(1) + i\sigma^{(ab;\tau)}(2). \qquad (42)$$

The saddle–point equations for the index combination $((a-1)a)$ are given by

$$\Sigma^{((a-1)a;\tau)} = \frac{2i}{N(-1,0)} \, \text{tr}\left[\left(\frac{1}{\mathbf{N}'(0)}\right)_{((a-1)a|aa)} \lambda C^{(1\tau)}(-1,0;r)\right],$$

$$(\Sigma^{((a-1)a;\tau)})^\dagger = \frac{2i}{N(-1,0)} \, \text{tr}\left[\left(\frac{1}{\mathbf{N}'(0)}\right)_{(aa|(a-1)a)} \lambda C^{(1\tau)}(-1,0;l)\right]. \quad (43)$$

Corresponding equations are obtained for $((a-1)(a+1))$ and $a(a+1)$. We test these results by adding Eq. (40) and the first of Eqs. (41), and the second and third of Eqs. (41). With the help of Eq. (25), this yields

$$\lambda\sigma^{(aa)}\delta_{\mu\nu} + \sum_\tau \lambda\sigma^{(aa;\tau)}C_{\mu\nu}^{(1\tau)}(0) = \frac{\lambda^2}{l}\sum_{\rho\sigma}\mathcal{A}_{\mu\nu;\rho\sigma}^{(0)}\left(\frac{1}{\mathbf{N}'(0)}\right)_{(aa|aa);\sigma\rho},$$

$$\lambda\sigma^{(a(a+1))}\delta_{\mu\nu} + \sum_\tau \lambda\sigma^{(aa;\tau)}C_{\mu\nu}^{(1\tau)}(1) = \frac{\lambda^2}{l}\sum_{\rho\sigma}\mathcal{A}_{\mu\nu;\rho\sigma}^{(1)}\left(\frac{1}{\mathbf{N}'(0)}\right)_{(aa|aa);\sigma\rho}.$$

$$(44)$$

Similarly, we find

$$-\frac{i}{2}\sum_\tau \lambda C_{\mu\nu}^{(1\tau)}(-1,0;l)\Sigma^{((a-1)a;\tau)} = \frac{\lambda^2}{l}\sum_{\rho\sigma}\mathcal{A}_{\mu\nu;\rho\sigma}^{(-1,0)}\left(\frac{1}{\mathbf{N}'(0)}\right)_{((a-1)a|aa);\sigma\rho},$$

$$\frac{i}{2}\sum_\tau \lambda C_{\mu\nu}^{(1\tau)}(-1,0;r)(\Sigma^{((a-1)a;\tau)})^\dagger = \frac{\lambda^2}{l}\sum_{\rho\sigma}\mathcal{A}_{\mu\nu;\rho\sigma}^{(0,-1)}$$

$$\times \left(\frac{1}{\mathbf{N}'(0)}\right)_{((aa|(a-1)a);\sigma\rho}, \qquad (45)$$

and completely analogous relations for the index combinations $((a-1)(a+1))$ and $(a(a+1))$. Eqs. (44) and (45) provide a test for our calculation in the

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following sense. On the left–hand sides of these equations, there occur the same linear combinations of the σ–matrices as in the matrix $\mathbf{N}'(0)$, see Eqs. (32) through (39). On the right–hand side, we encounter traces over the inverse of $\mathbf{N}'(0)$ times the corresponding matrix element of the second moment of H_0. This is exactly the same structure as encountered by Benet $et\ al.$ [11]. We note that using the orthonormality relations of the coefficients $C^{(s\tau)}$, we can retrieve the original saddle–point equations [Eqs. (40) through (43)] from Eqs. (44) and (45). Thus, the latter set of equations encapsulates the saddle–point conditions.

The structure of the saddle–point equations is displayed most clearly when we introduce the following notation. For the matrix $\mathbf{N}'(0)$, we write

$$\langle ab\mu|\mathbf{N}'(0)|a'b'\nu\rangle = (E\delta_{\mu\nu} - w_{\mu\nu}\delta_{ab})\delta_{aa'}\delta_{bb'} - \langle ab\mu|X|a'b'\nu\rangle \ . \qquad (46)$$

By definition, the matrix X contains all the σ–matrices which occur in $\mathbf{N}'(0)$. Needless to say that the possible index combinations (a,b) and (a',b') are severely restricted by our choice of admissible Hilbert vectors. Using this notation and the definitions of the matrices $\mathcal{A}$ in Section 3, we can cast the totality of saddle–point equations in the form

$$\langle ab\mu|X|a'b'\nu\rangle = \sum_{\rho\sigma} \overline{\langle ab\mu|H_0|ab\sigma\rangle\left(\frac{1}{E-w-X}\right)_{(ab|a'b');\sigma\rho}\langle a'b'\rho|H_0|a'b'\nu\rangle} \ . \tag{47}$$

Eq. (47) is the generalization of the Pastur equation to the present problem.

6 Solution of the Saddle–Point Equations

We follow Ref. [11] and solve the saddle–point equations by iteration. In Eq. (47), iteration generates a continued–fraction expansion which we break off after n steps where n is integer but arbitrary. The nth element of the continued–fraction expansion has the form

$$\sum_{\rho\sigma} \overline{\langle ab\mu|H_0|ab\sigma\rangle\left(\frac{1}{E-w}\right)_{(ab|a'b');\sigma\rho}\langle a'b'\rho|H_0|a'b'\nu\rangle} \ . \tag{48}$$

But $(E-w)$ is diagonal in the box indices (ab) and $(a'b')$. Therefore, expression (48) equals

$$\sum_{\rho\sigma} \overline{\langle ab\mu|H_0|ab\sigma\rangle\left(\frac{1}{E-w}\right)_{(ab|ab);\sigma\rho}\langle ab\rho|H_0|ab\nu\rangle} \ \delta_{aa'}\delta_{bb'} \ . \tag{49}$$

Using this result in the $(n-1)^{\text{st}}$ element of the continued–fraction expansion, we obtain a denominator which once again is diagonal in the box indices (ab) and $(a'b')$. Continuing the argument all the way up to the first term, we conclude that all contributions to X which are not diagonal in the box indices, vanish. Thus, we have for $i = 1, 2$

$$\sigma_{\text{sp}}^{((a-1)a;\tau)}(i) = 0 \ ; \ \sigma_{\text{sp}}^{((a-1)(a+1);\tau)}(i) = 0 \ ; \ \sigma_{\text{sp}}^{(a(a+1);\tau)}(i) = 0 \ . \tag{50}$$

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The index sp stands for saddle point. It follows that the saddle–point equations separate into a set of $(2K-1)$ equations for the box–diagonal elements of X. For $(ab) = (a(a+1))$ we have $w = 0$, and the nth element of the continued–fraction expansion is given by

$$\sum_{\rho\sigma} \overline{\langle a(a+1)\mu|H_0|a(a+1)\sigma\rangle E^{-1}\delta_{\sigma\rho}\langle a(a+1)\rho|H_0|a(a+1)\nu\rangle}$$

$$= \frac{\lambda^2}{l}\Lambda^{(0)}(1)\delta_{\mu\nu} . \tag{51}$$

The right–hand side of this equation follows from Eq. (20) and from the fact that $C_{\mu\nu}^{(1\tau)}(-1)$ is traceless. The fact that E is multiplied by the unit matrix $\delta_{\mu\nu}$ implies the same statement for the nth element of the continued–fraction expansion. Again, we can continue the argument all the way up to the first term and conclude that

$$\sigma_{\text{sp}}^{(a(a+1);\tau)} = 0 . \tag{52}$$

Therefore, the only remaining matrix is $\sigma^{(a(a+1))}$. We use Eqs. (11) and (18), the fact that in the latter only the term with $s = 0$ is relevant, and the fact that $C_{\mu\nu}^{(0)} = \delta_{\mu\nu}$. It follows that $\sigma^{(a(a+1))}$ obeys the standard saddle–point equation

$$\sigma_{\text{sp}}^{(a(a+1))} = \frac{\lambda}{E - \lambda\sigma_{\text{sp}}^{(a(a+1))}} . \tag{53}$$

As usual, the solution is obtained in two steps. First, we determine the diagonal elements $\sigma_{\text{d}}^{(a(a+1))}$ which obey Eq. (53). For $|E| \leq 2\lambda$, these are given by

$$\sigma_{\text{d}}^{(a(a+1))} = \frac{E}{2\lambda} \pm i\Delta_1(E) \tag{54}$$

where we have defined

$$\Delta_1(E) = \sqrt{1 - \left(\frac{E}{2\lambda}\right)^2} . \tag{55}$$

We observe that the quantity $\Delta_1(E)$ is proportional to the average level density $\rho_{\text{sp1}}(E)$. Eq. (55) shows that $\rho_{\text{sp1}}(E)$ has the shape of a semicircle with radius 2λ. The invariance of the saddle–point equation Eq. (54) under pseudounitary graded transformations T implies that the saddle–point manifold is given by [16]

$$\sigma_{\text{sp}}^{(a(a+1))} = T^{(a(a+1))} \left[\frac{E}{2\lambda} - i\ \Delta_1(E)\ L \right] (T^{(a(a+1))})^{-1}$$

$$= \frac{E}{2\lambda} - i\ \Delta_1(E)\ T^{(a(a+1))}L(T^{(a(a+1))})^{-1} . \tag{56}$$

For the index combination (aa), the saddle–point equation contains the two–body interaction w. We are interested in comparing the cases $w = 0$ and $w \neq 0$ and, therefore, treat both cases. For $w = 0$, we conclude as before that

$$\sigma_{\text{sp}}^{(aa;\tau)} = 0 \text{ for } w = 0 . \tag{57}$$

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For $\sigma^{(aa)}$, we obtain the same saddle–point equation [Eq. (53)] as for $\sigma^{(a(a+1))}$. Eq. (40) and the second of Eqs. (41) actually differ by a factor 2. The factor disappears, however, when we take the traces over $\delta_{\rho\sigma}$ because the dimensions N_{aa} and $N_{a(a+1)}$ differ by that same factor. The solution is accordingly given by

$$
\begin{aligned}
\sigma_{\mathrm{sp}}^{(aa)} &= T^{(aa)} \left[\frac{E}{2\lambda} - i \sqrt{1 - \left(\frac{E}{2\lambda}\right)^2} \, L \, \right] (T^{(aa)})^{-1} \\
&= \frac{E}{2\lambda} - i \sqrt{1 - \left(\frac{E}{2\lambda}\right)^2} \, T^{(aa)} \, L \, (T^{(aa)})^{-1} \text{ for } w = 0 \; . \quad (58)
\end{aligned}
$$

For $w \neq 0$, we encounter a novel situation because $w_{\mu\nu}$ is a genuine matrix and the arguments used above do not apply. The nth element of the continued–fraction expansion has the form

$$
\sum_{\rho\sigma} \overline{\langle aa\mu|H_0|aa\sigma\rangle \left(\frac{1}{E-w}\right)_{(aa|aa);\sigma\rho} \langle aa\rho|H_0|aa\nu\rangle} \; . \quad (59)
$$

The second moment of H_0 in expression (59) is proportional to the matrix $\mathcal{A}^{(0)}$ in Eq. (12) for which we use the eigenvalue expansion Eq. (25). We first consider the terms with $s = 1$ for which the sum over (ρ, σ) takes the form $\sum_{\rho\sigma} C_{\rho\sigma}^{(1\tau)}(0)(1/[E-w])_{(aa|aa);\sigma\rho}$ with $C_{\mu\nu}^{(1\tau)}(0) \propto \langle aa\mu|\alpha_m^\dagger \alpha_n|aa\nu\rangle$ (where $m \neq n$) or $C_{\mu\nu}^{(1\tau)}(0) \propto \langle aa\mu|(\alpha_m^\dagger \alpha_m - (1/l)\sum_n \alpha_n^\dagger \alpha_n)|aa\nu\rangle$. We write $M_{\mu\nu} = ((E - w)^{-1})_{\mu\nu}$. For $C_{\mu\nu}^{(1\tau)}(0) \propto \langle aa\mu|\alpha_m^\dagger \alpha_n|aa\nu\rangle$, we find $\sum_{\rho\sigma} C_{\rho\sigma}^{(1\tau)}(0)M_{\mu\nu} = (4/l^2)\sum_j M_{njmj}$ where we have returned to the original notation in Eq. (8). Expanding M in powers of w and taking the linear term, we note that the matrix element w_{njmj} vanishes for $m \neq n$. This can easily be seen by using in each slice periodic boundary conditions in longitudinal direction and follows from momentum conservation. The same statement applies to all higher moments of w. We conclude that $\sum_{\rho\sigma} \langle aa\rho|\alpha_m^\dagger \alpha_n|aa\sigma\rangle M_{\sigma\rho} = 0$. We turn to $C_{\mu\nu}^{(1\tau)}(0) \propto \langle aa\mu|(\alpha_m^\dagger \alpha_m - (1/l)\sum_n \alpha_n^\dagger \alpha_n)|aa\nu\rangle$ and find that $\sum_{\rho\sigma} C_{\rho\sigma}^{(1\tau)}(0)M_{\mu\nu}$ is proportional to $(4/l^2)(\sum_j M_{mjmj} - (1/l)\sum_{jn} M_{njnj})$. We again consider first the term linear in w. Although we do not expect the condition $w_{mjmj} = w_{njnj}$ to hold for arbitrary values of m and n, we believe that averaging over j leads to an approximate fulfillment of the condition $\sum_j w_{mjmj} = \sum_j w_{njnj}$ for arbitrary values of m and n. This is expected to be true *a fortiori* for higher powers of w. Thus, in very good approximation the terms with $s = 1$ do not contribute to the expression (59), the argument applies equally to all terms in the continued–fraction expansion, and we obtain

$$
\sigma_{\mathrm{sp}}^{(aa;\tau)} \approx 0 \text{ for } w \neq 0 \; . \quad (60)
$$

We are left with $\sigma^{(aa)}$ which obeys the saddle–point equation

$$
\sigma_{\mathrm{sp}}^{(aa)} = N_{aa}^{-1} \, \mathrm{tr}[(\frac{\lambda}{E - w - \lambda\sigma_{\mathrm{sp}}^{(aa)}})_{\mu\nu}] \; . \quad (61)
$$

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This is easily verified by comparing the continued–fraction expansion of Eq. (61) with that of Eq. (49) for $a = b$ after we drop in the latter the terms with $s = 1$. We recall that $N_{aa} = l(l-1)/2$. The matrix w is Hermitean. We denote the eigenvalues of w by ε_j with $j = 1, \ldots, N_{aa}$. Without loss of generality, we choose $\varepsilon_1 \le \varepsilon_2 \le \ldots \le \varepsilon_{N_{aa}}$. The saddle–point equation Eq. (61) takes the form

$$\sigma_{\rm sp}^{(aa)} = N_{aa}^{-1} \sum_j \frac{\lambda}{E - \varepsilon_j - \lambda \sigma_{\rm sp}^{(aa)}} \ . \tag{62}$$

As is the case for $w = 0$, this equation is invariant under all graded pseudounitary transformations $T^{(aa)}$. Therefore, we first determine the scalar solutions $\sigma_{\rm d}^{(aa)}$ and from there construct the saddle–point manifold as in Eq. (58). Eq. (62) can be written as a polynomial of order N_{aa} in $\sigma_{\rm d}^{(aa)}$ and, therefore, possesses N_{aa} real or complex solutions. We show that of these, $N_{aa} - 2$ are always real. The remaining two solutions are either real or complex conjugate to each other. The positive imaginary part determines the level density $\rho_{\rm sp0}(E)$ in the box (aa). Therefore, the spectrum extends over that energy interval where two complex conjugate solutions of Eq. (62) exist. We determine the spectrum geometrically.

As a preparatory step, we return to Eq. (53). With $z = E/\lambda$ and $\tau = z - \sigma_{\rm d}^{a(a+1)}$, we write this equation in the form

$$z - \tau = 1/\tau \ . \tag{63}$$

The right–hand side is a hyperbola with asymptotes along the abscissa and the ordinate. The left–hand side represents a bundle of straight lines which intersect the abscissa at an angle of $-\pi/4$. For sufficiently large values of $|E|$, each of these straight lines intersects the hyperbola twice while for small values of $|E|$, there are no points of intersection. The end points of the spectrum coincide with the points where a straight line osculates the hyperbola, i.e., where the derivative of the hyperbola equals -1. This is the case at $\tau = \pm 1$ or, with Eq. (63), where $z = \pm 2$ and, thus, $E = \pm 2\lambda$. This agrees with Eq. (54).

We apply the same consideration to Eq.(62) and again use $z = E/\lambda$ and $\tau = z - \sigma_{\rm d}^{(aa)}$. Then,

$$z - \tau = N_{aa}^{-1} \sum_j \frac{1}{\tau - \varepsilon_j/\lambda} \ . \tag{64}$$

Each term on the right–hand side of this equation is a hyperbola with asymptotes along the abscissa and along a straight line parallel to the ordinate which intersects the abscissa at $\tau = \varepsilon_j/\lambda$. The left–hand side again represents a bundle of straight lines which intersect the abscissa at an angle of $-\pi/4$. Figure 1 represents the situation schematically. We see that for sufficiently large values of $|E|$, each straight line intersects the sum of the hyperbolas exactly N_{aa} times, giving rise to N_{aa} real solutions. For sufficiently small values of $|E|$, the number of intersections is $N_{aa} - 2$. The spectrum extends continuously from a point E_1 to the left of ε_1 to a point E_2 to the right of $\varepsilon_{N_{aa}}$. Again, the energies $E_{1,2}$ are defined in terms of τ_1 and τ_2. The latter are those points where one of the

straight lines $z - \tau$ osculates the sum of the hyperbolas, i.e., where the gradient of the latter is -1. With increasing strength of the two–body interaction w, the difference $\varepsilon_{Naa} - \varepsilon_1$ grows monotonically and so does, therefore, the difference $E_2 - E_1$. This difference gives the width of the spectrum in the box (aa). Thus, the Wigner semicircle appearing in Eq. (54) is gradually deformed by becoming wider and flatter (the area remains the same).

We supplement these general considerations by calculating position and width of the spectrum perturbatively for small values of w. It is easily seen that the first–order term only shifts the center of the semicircle from zero to $\mathrm{tr}(w)/N_{aa}$, without changing either radius or shape of the semicircle. We accordingly define $E_0 = \mathrm{tr}(w)/N_{aa}$ and $z_0 = E_0/\lambda$. Introducing the normalized variance of w as $U^2 = \{(1/N_{aa})\mathrm{tr}(w^2) - [(1/N_{aa})\mathrm{tr}(w)]^2\}/\lambda^2$ and expanding the right–hand side of Eq. (64) in powers of w, we get

$$z - z_0 - \tau = \frac{1}{\tau} + \frac{U^2}{\tau^3} + \dots \; . \tag{65}$$

To lowest order in U^2, the points τ_1 and τ_2 are given by $\tau_{1,2} = \pm(1 + (3/2)U^2)$. The end points $E_{1,2}$ of the spectrum are accordingly given by

$$E_{1,2} = E_0 \pm (2 + U^2)\lambda \; . \tag{66}$$

This result clearly shows the widening of the spectrum.

How strong a two–body interaction is needed to yield a sizeable effect on the localization length? A qualitative change of the spectrum in the box (aa) and, thus, in localization occurs whenever $E_{1,2}$ and, thus, E_0 and U^2/λ are of the order of λ, the width of the unperturbed two–body spectrum in box (aa). In order to be independent of the sign of w, we focus attention on one contribution to U^2, i.e., the term $(1/N_{aa})\mathrm{tr}(w^2)/\lambda^2$ which must then be of order unity. For realistic two–body interactions, the expression $\mathrm{tr}(w^2)$ may not exist. Therefore, we use completeness and write $(1/N_{aa})\mathrm{tr}(w^2)$ in the form $\langle aa\mu|w^2|aa\mu\rangle_{\mathrm{av}}$ where the index av stands for an average over the states labelled μ. We obtain $\langle aa\mu|w^2|aa\mu\rangle_{\mathrm{av}} = \lambda^2$. To eliminate λ, we use Eq. (5) which yields $\lambda^2 = \sum_j (h_{ij}^{(a)})^2$. We identify the matrix elements $h_{ij}^{(a)}$ with the matrix elements $\langle ai|V_{\mathrm{imp}}|aj\rangle$ of the impurity potential V_{imp}. We do so because upon discretization the kinetic (potential) terms of a continuum model turn into hopping matrix elements (local energies, respectively). We replace the ensemble average by a running average over the single–particle states, use completeness, and obtain

$$\langle aa\mu|w^2|aa\mu\rangle_{\mathrm{av}} \geq \langle ai|(V_{\mathrm{imp}})^2|ai\rangle_{\mathrm{av}} \; . \tag{67}$$

Whenever the criterion (67) is met, the two–body interaction will change the localization length qualitatively. Quantitative changes set in for smaller values of w^2, of course. Because of the normalization of the wave functions, the criterion (67) is essentially independent of the size of the slices. This is physically reasonable. Moreover, the criterion (67) can easily be checked in concrete cases. Since the screened Coulomb interaction is more or less fixed, the criterion (67)

effectively establishes an upper bound on the strength of the impurity potential for interaction effects to be relevant. The criterion (67) has only qualitative significance because it is based upon the saddle–point approximation. When we identify the left–hand side with the mean–square matrix element U^2 in Eq. (1) and the left–hand side with the bandwidth B^2 we see the close correspndence between our criterion and the result Eq. (1). We point out, however, that on the quantitative level, our result differs from Eq. (1). This is discussed in Sections 9 and 10.

Returning to the general case, we write $\sigma_{\rm d}^{(aa)}$ in the form

$$\sigma_{\rm d}^{(aa)} = a(E) \pm i\Delta_0(E) \tag{68}$$

where $\Delta_0(E) > 0$ for $E_1 < E < E_2$ is proportional to the spectral density $\rho_{\rm sp0}(E)$. The general form of the saddle–point solution is then

$$\sigma_{\rm sp}^{(aa)} = T^{(aa)} \left[\; a(E) - i\Delta_0(E)\; L \;\right] (T^{(aa)})^{-1}$$
$$= a(E) - i\Delta_0(E)\; T^{(aa)} L (T^{(aa)})^{-1} \;. \tag{69}$$

The essential difference between the saddle–point solutions in Eqs. (69) and (56) lies in the difference between $\Delta_0(E)$ and $\Delta_1(E)$, i.e, in the different spectral densities $\rho_{\rm sp0}(E)$ and $\rho_{\rm sp1}(E)$. Only for $w = 0$ do we have $\Delta_0(E) = \Delta_1(E)$.

7 Non–Linear Sigma Model. Localization Properties

We now expand the effective Lagrangean in the exponent of Eq. (31) around the saddle–point solutions obtained in Section 6. To this end, we write for $i = 1, 2$

$$
\begin{aligned}
\sigma^{(aa)} &= \sigma_{\rm sp}^{(aa)} + T^{(aa)} \delta P^{(aa)} (T^{(aa)})^{-1} \;, \\
\sigma^{(a(a+1))} &= \sigma_{\rm sp}^{(a(a+1))} + T^{(a(a+1)} \delta P^{(a(a+1)} (T^{(a(a+1))})^{-1} \;, \\
\sigma^{(aa;\tau)} &= T^{(aa)} \delta\sigma^{(aa;\tau)} (T^{(aa)})^{-1} \;, \\
\sigma^{(a(a+1);\tau)} &= T^{(a(a+1)} \delta\sigma^{(a(a+1);\tau)} (T^{(a(a+1))})^{-1} \;, \\
\sigma^{((a-1)a;\tau)}(i) &= T^{((a-1)a)} \delta\sigma^{((a-1)a;\tau)}(i) (T^{(aa)})^{-1} \;, \\
\sigma^{((a-1)(a+1);\tau)}(i) &= T^{((a-1)a)} \delta\sigma^{((a-1)(a+1);\tau)}(i) (T^{(a(a+1))})^{-1} \;, \\
\sigma^{(a(a+1);\tau)}(i) &= T^{(aa)} \delta\sigma^{(a(a+1);\tau)}(i) (T^{(a(a+1))})^{-1} \;.
\end{aligned} \tag{70}
$$

The quantities $\delta P^{(aa)}$ and $\delta P^{(a(a+1)}$ are block–diagonal in a representation of the supermatrices where the first (second) block of dimension two corresponds to the first (the second) Green's function, respectively, see Ref. [16]. In writing Eqs. (70), we have used the freedom to multiply each $\delta\sigma$ and each δP from the left (right) by the corresponding matrix T (T^{-1}, respectively). The volume of integration $\mathrm{d}[\sigma]$ in Eq. (31) changes into the product of the integration measures for all the T's, δP's, and $\delta\sigma$'s. Substituting Eqs. (70) into Eq. (31) and expanding the exponent in powers of the δP's and $\delta\sigma$'s, we generate terms of order $0, 2$

and $t \geq 3$. (The linear terms vanish approximately because of the saddle–point condition, approximately because of Eq. (60)). In the present Section, we focus attention on the terms of order zero. The remaining terms give rise to the loop expansion. This expansion is investigated in Sections 8 and 9 below.

In addition to the $\delta\sigma$'s and δP's, $\mathbf{N}(J)$ also contains v and J, and we must clarify how to handle contributions arising from these terms. We first address the choice of J which is needed for investigating localization. We consider the average two–point function $C(n)$ defined in Eq. (9). If the system is localized, this quantity decays for $n \gg 1$ exponentially with increasing n. The exponential decay is governed by the localization length. In the expression (9), the label a need not be equal to 1 nor do we put $(a + n + 1)$ equal to K, the total number of slices. In fact, choosing $1 \ll a \ll (a + n) \ll K$ is expected to display the localization properties without edge effects. Instead of the Green's functions in expression (9), we might have considered another expression where the bra state (ket state) is chosen as $\langle aa|$ (as $|(a + n)(a + n)|\rangle$, respectively). Localization properties should not depend upon this choice, and this is indeed what we shall find. To generate the expression (9) from the source terms, we choose the matrix J to have only the following non–zero matrix elements.

$$\langle a(a + 1)\mu|J|(a + n)(a + n + 1)\nu\rangle = \delta_{\mu\mu_0}\delta_{\nu\nu_0}k_{\alpha\beta}j_{\alpha\beta}$$
$$\langle (a + n)(a + n + 1)\nu|J|a(a + 1)\mu\rangle = \delta_{\mu\mu_0}\delta_{\nu\nu_0}k_{\alpha\beta}j_{\alpha\beta} \ . \tag{71}$$

Here k is a four–by–four graded diagonal matrix with diagonal matrix elements $+1$ (-1) in the Boson–Boson block (the Fermion–Fermion block, respectively), and j is a four–by–four graded diagonal matrix with diagonal matrix elements j_1 (j_2) for the retarded (advanced) Green's function, respectively. The term (9) is obtained by differentiating $\overline{Z}$ once with respect to both j_1 and j_2 at $j_1 = 0 = j_2$.

In order to illuminate the role of the hopping terms v, it is instructive to display the structure of the source terms for the case of the saddle–point solution, i.e., when all the $\delta\sigma$'s and δP's are put equal to zero. We define

$$
\begin{aligned}
g_{\mu\nu}^{(a(a+1))} &= T^{(a(a+1))} \, g_{\mathrm{d};\mu\nu}^{(a(a+1))} \, (T^{(a(a+1))})^{-1} \\
&= T^{(a(a+1))} \, \delta_{\mu\nu} \, [\frac{E}{2\lambda} - i\Delta_1 L]^{-1} \, (T^{(a(a+1))})^{-1} \ , \\
g_{\mu\nu}^{(aa)} &= T^{(aa)} \, g_{\mathrm{d};\mu\nu}^{(aa)} \, (T^{(aa)})^{-1} \\
&= T^{(aa)} \, ([\frac{E}{2\lambda} - a(E) - i\Delta_0 L) - w]^{-1})_{\mu\nu} \, (T^{(aa)})^{-1} \ . \tag{72}
\end{aligned}
$$

We put $v = 0$ and expand tr trg $\ln \mathbf{N}(J)$ in powers of J, keeping terms of orders 1 and 2 which we denote by $(0, 1)$ and $(0, 2)$, respectively. We expand the exponential in powers of J and keep the terms of second order, given by $(0, 1)^2$ and by $(0, 2)$. The term $(0, 1)$ actually vanishes and the term $(0, 2)$ is proportional to

$$\mathrm{trg} \sum_{\mu\nu\rho\sigma} g_{\mu\nu}^{(a(a+1))} \, \langle a(a + 1)\nu|J|(a + n)(a + n + 1)\rho\rangle$$

$$\times g_{\rho\sigma}^{((a+n)(a+n+1))} \, \langle (a + n)(a + n + 1)\sigma|J|a(a + 1)\mu\rangle \ . \tag{73}$$

This is the standard term used in the calculation of localization properties using the non–linear sigma model [13, 14]. It is easy to see that there are non–vanishing contributions from higher orders in v/λ. Since $v \ll \lambda$, such contributions can be omitted.

It remains to work out the terms proportional to $(v/\lambda)^2$. We have already calculated the source terms and, therefore, put $J = 0$. The matrix $\mathbf{N}_{\mathrm{sp}}(0)$ has the following non–zero matrix elements.

$$\langle aa\mu|\mathbf{N}_{\mathrm{sp}}(0)| \ aa\nu \rangle = \left(E - \lambda\sigma_{\mathrm{sp}}^{(aa)} \right) \delta_{\mu\nu} - w_{\mu\nu} \ ,$$

$$\langle a(a+1)\mu|\mathbf{N}_{\mathrm{sp}}(0)| \ a(a+1)\nu \rangle = \left(E - \lambda\sigma^{(a(a+1))} \right) \delta_{\mu\nu} \ ,$$

$$\langle (a-1)a\mu|\mathbf{N}_{\mathrm{sp}}(0)|aa\nu \rangle = \langle aa\mu|\mathbf{N}_{\mathrm{sp}}(0)|(a-1)a\nu \rangle = -v \ \delta_{\mu\nu} \ ,$$

$$\langle aa\mu|\mathbf{N}_{\mathrm{sp}}(0)|a(a+1)\nu \rangle = \langle a(a+1)\mu|\mathbf{N}_{\mathrm{sp}}(0)|aa\nu \rangle = -v \ \delta_{\mu\nu} \ . \quad (74)$$

We use the fact that $\sigma_{\mathrm{sp}}^{(aa)}$ and $\sigma_{\mathrm{sp}}^{(a(a+1))}$ are solutions of the saddle–point Eqs. (62) and (53). Moreover, both the matrix v and $\sigma_{\mathrm{sp}}^{(a(a+1))}$ are proportional to $\delta_{\mu\nu}$. This yields

$$-\mathrm{trg}\ \mathrm{tr}\ln\mathbf{N}_{\mathrm{sp}}(0) = +(v/\lambda)^2 \sum_a [(N_{aa} + N_{a(a+1)})/2]$$

$$\times [\mathrm{trg}(\sigma_{\mathrm{sp}}^{(aa)}\sigma_{\mathrm{sp}}^{(a(a+1))}) + \mathrm{trg}(\sigma_{\mathrm{sp}}^{(a(a+1)}\sigma_{\mathrm{sp}}^{((a+1)(a+1))})] \ . \quad (75)$$

The right–hand side of Eq. (75) contains the source terms (proportional to powers of J) and, most importantly, the term responsible for electron transport through the wire. With Eqs. (56) and (69) and the explicit values of N_{aa} and $N_{a(a+1)}$, this last term has the form

$$+(v/\lambda)^2(3l^2/4)\Delta_0\Delta_1 \left(\sum_a \mathrm{trg}(T^{(aa)}L(T^{(aa)})^{-1}T^{(a(a+1))}L(T^{(a(a+1))})^{-1}) \right.$$

$$\left. + \sum_a \mathrm{trg}(T^{(a(a+1))}L(T^{(a(a+1))})^{-1}T^{((a+1)(a+1))}L(T^{((a+1)(a+1))})^{-1}) \right) .(76)$$

We recall that we are interested in comparing electron transport through the wire for the cases $w = 0$ and $w \neq 0$. We note that in the formulation of Eq. (76), the only difference between the two cases lies in the difference of the values of $\Delta_0(E)$. For $w = 0$ we have $\Delta_0(E) = \Delta_1(E) = \sqrt{1 - (E/2\lambda)^2}$ while this equality is violated for $w \neq 0$.

Eq. (76) establishes a non–linear sigma model for the transport of two electrons through a quasi one–dimensional wire. This model is equivalent to the model for the transport of a single electron through the same wire studied in Ref. [12]. To see this, we recall that the model of Ref. [12] also considered the wire as divided into K' slices numbered $j = 1, \ldots, K'$. We put $K' = 2K - 1$ and map the $2K - 1$ boxes labelled (aa) and $a(a+1)$ of the two–electron problem onto the K' slices of the single–electron problem of Ref. [12] by putting

$j = 2a - 1$ for the boxes (aa) and $j = 2a$ for the boxes $(a(a + 1))$. Then, the terms in Eq. (76) take the form

$$+ (v/\lambda)^2 (3l^2/4) \Delta_0 \Delta_1 \sum_{j=1}^{K'} \mathrm{trg}(T^{(j)} L (T^{(j)})^{-1} T^{(j+1)} L (T^{(j+1)})^{-1}) \ . \qquad (77)$$

This is exactly the form of the non–linear sigma model derived for the transport of a single electron through a disordered wire in Ref. [12]. A similar correspondence can be established for the source terms. The localization properties of the non–linear sigma model Eq. (77) for single electron transport were extensively studied in Refs. [13, 14]. The correspondence between the models in Eqs. (77) and (76) makes it possible to use these results for the discussion of the localization properties of the model of two interacting electrons in Eq. (76). All we have to do is to transcribe the results of Refs. [13, 14] into the present framework.

In Refs. [13, 14], the continuum limit was taken by letting the length of each slice go to zero and the number K' of slices go to infinity. The same must be done in the framework of Eq. (76). We take this step in Section 9 below. At the moment, it suffices to observe that the localization length is directly proportional to the coefficient $(v/\lambda)^2 (3l^2/4) \Delta_0 \Delta_1$ appearing in Eq. (77). This statement then holds likewise for Eq. (76). We recall that Δ_0 and Δ_1 are proportional to the spectral densities $\rho_{\mathrm{sp0}}(E)$ and $\rho_{\mathrm{sp1}}(E)$, respectively, as determined by the saddle–point condition. We accordingly define the saddle–point approximation T_{sp} to the transport coefficient (sometimes also referred to as the microscopic dimensionless conductance) as

$$T_{\mathrm{sp}} = 2\pi \ \rho_{\mathrm{sp0}}(E) \ v^2 \ \rho_{\mathrm{sp1}}(E) \ . \qquad (78)$$

This definition is patterned after the general definition of transport coefficients for stochastic quantum problems in Ref. [17]. Transport is mediated by the strength v^2 of the hopping matrix elements connecting neighboring groups of states. The dimensionless transport coefficient is symmetric with respect to the interchange of the two boxes (aa) and $(a(a + 1))$. Transport is possible only if the product $\rho_{\mathrm{sp0}}(E) \ \rho_{\mathrm{sp1}}(E)$ differs from zero. Our result (79) is subject to this constraint. Let $\Delta E_1 \ (\Delta E_0(w))$ denote the energy interval where $\rho_{\mathrm{sp1}}(E) \neq 0 \ (\rho_{\mathrm{sp0}}(E) \neq 0$, respectively). We obviously have $\Delta E_0(w = 0)) = \Delta_1$. Typically we expect that for $w \neq 0$ the interval $\Delta E_0(w))$ is both shifted and widened in comparison with ΔE_1, the widening outweighing the shift. In this case, the energy interval where electron transport is possible is given by $\Delta E_0(w = 0)) = \Delta E_1$.

The ratio of the localization lengths $\zeta(w \neq 0)$ for non–vanishing two–body interaction and $\zeta(w = 0)$ for vanishing two–body interaction, is accordingly given by

$$\frac{\zeta(w \neq 0)}{\zeta(w = 0)} = \frac{\rho_{\mathrm{sp0}}(E)}{\rho_{\mathrm{sp1}}(E)} \ . \qquad (79)$$

Eq. (79) is valid within the same energy interval as the result (78) for the transport coefficient. The ratio (79) should typically be larger (smaller) than unity at the edges (in the middle) of the spectrum.

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While physically very plausible, the form of Eq. (76) has one obvious flaw. The transport coefficient is proportional to the approximate level densities as determined by the saddle–point approximation rather than to the exact level densities. And we know for sure that at least $\rho_{\mathrm{sp1}}(E)$ (which has the shape of a semicircle) differs from the exact level density. The latter is the level density of two non–interacting electrons each subject to a random Hamiltonian and, therefore, given by the convolution of two one–body densities of semicircle shape. This yields a non–zero density in the interval $[-4\lambda, 4\lambda]$ the shape of which is intermittent between a semicircle and a Gaussian. We conclude that Eq. (76) cannot possibly be the exact answer. This is why we turn now to a study of the loop expansion (the expansion of the integrand of the generating function around the saddle–point solution).

8 Loop Expansion: Terms of order zero in v

We investigate the terms in the loop expansion of $\overline{Z}$. We recall that this expansion is generated by expanding the exponent in Eq. (31) in powers of the $\delta\sigma$'s and δP's, by expanding the exponential containing terms of higher order than the second in a Taylor series, and by carrying out the resulting Gaussian integrations.

Expanding the quadratic terms in the exponent in Eq. (31) in powers of the $\delta\sigma$'s and δP's is trivial. Terms of zeroth order vanish. Terms of first order cancel against those stemming from the expansion of $\mathrm{tr\ trg\ ln\,\mathbf{N}}(J)$. The terms of second order are

$$
\begin{aligned}
-\sum_a \mathrm{trg}\bigg(& \frac{l^2}{8}\,(\delta P^{(aa)})^2 + \frac{l^2}{4}\,(\delta P^{(a((a+1))})^2 \\
& + \sum_\tau \bigg\{ \frac{l^2}{4}\,(\delta\sigma^{(aa;\tau)})^2 + \frac{l^2}{4}\,(\delta\sigma^{(a(a+1);\tau)})^2 \\
& + \sum_{i=1}^2 \big(\frac{N(-1,0)}{4}\,(\delta\sigma^{((a-1)a;\tau)}(i))^2 + \frac{N(-1,1)}{4}\,(\delta\sigma^{((a-1)(a+1);\tau)}(i))^2 \\
& + \frac{N(0,1)}{4}\,(\delta\sigma^{(a(a+1);\tau)}(i))^2 \big) \bigg\} \bigg)
\end{aligned}
\tag{80}
$$

Attention thus focusses on the term $-\mathrm{tr\ trg\ ln\,N}(J)$ as given by Eqs. (32) to (39), with the understanding that the substitutions (70) have been made.

We recall that we deal with the case $v \ll \lambda$. Indeed, in the calculation of localization properties, it is customary to consider only the zeroth order contribution in v/λ to the source term. In the calculation of the loop corrections, we follow this usage. It turns out that for a complete understanding of localization properties, terms of order v^2 are also important. These are studied in Section 9.

Thus, the loop expansion is generated by putting $v = 0$ in $\mathbf{N}(J)$ and expanding the logarithm both in powers of J and in powers of the $\delta\sigma$'s and δP's. We denote the terms generated in this way by (p, q) where p denotes the combined

power of the $\delta\sigma$'s and δP's, and where q denotes the power of J. We obviously need to consider only the cases where $q = 0, 1, 2$. The expansion is simplified because of the form of Eqs. (70) and (72): The factors T and T^{-1} cancel everywhere except in the source terms J, and the expansion proceeds effectively in powers of $g_\mathrm{d}\delta P$, $g_\mathrm{d}\delta\sigma$, and $g_\mathrm{d}T^{-1}JT$. Here the g_d's are defined in Eqs. (72).

We evaluate first the lowest–order terms and, later, turn to the general form of the loop expansion. We consider the terms (p, q) with $p \leq 4$ and $q \leq 2$. The term $(0, 0)$ vanishes identically. As remarked above, the term $(1, 0)$ approximately cancels against the linear terms stemming from the quadratic expressions in the exponent of the generating function. The terms $(p, 1)$ vanish for $p < n$. Upon expansion of the exponential and calculation of the Gaussian integrals, all terms which are odd in the $\delta\sigma$'s and δP's vanish. This leaves us with the following combinations of brackets (up to fourth order in total).

$$(0, 2) \; ; \; (2, 2) \; ; \; (4, 2) \; ; \; (1, 2)(3, 0) \; ; \; (2, 2)(2, 0) \; ; (0, 2)(2, 0) \; ;$$
$$(0, 2)(2, 0)(2, 0) \; ; \; (0, 2)(4, 0) \; . \tag{81}$$

In writing the terms $(2, 2)(2, 0)$ and $(0, 2)(2, 0)(2, 0)$ we imply that we also expand the second–order terms $(2, 0)$ originating from the expansion of the logarithm into a Taylor series. This procedure differs from the standard loop expansion where such second–order terms are added to the ones which stem from the quadratic terms in the exponent of Eq. (70). We do so for technical reasons. We shall see later that this step does not affect our conclusions. The term $(0, 2)$ was calculated in Section 7. We focus attention on the remaining terms.

It is useful to work out the Gaussian integrals first (prior to taking the traces in Hilbert space). The Gaussian integrals can be performed using Wick contraction [18]. We use that

$$\begin{aligned} \overline{\mathrm{trg}(\delta\sigma A)\mathrm{trg}(\delta\sigma B)} &= \frac{1}{N}\mathrm{trg}(AB) \; , \\ \overline{\mathrm{trg}(\delta\sigma A\delta\sigma B)} &= \frac{1}{N}\mathrm{trg}(A)\mathrm{trg}(B) \; . \end{aligned} \tag{82}$$

These rules apply for arbitrary graded matrices A and B and can easily be checked by direct calculation. The symbol $\delta\sigma$ stands for δP or for any of the $\delta\sigma$'s, and N is any of the large factors multiplying the quadratic terms in Eq. (80). Needless to say, the pair of $\delta\sigma$'s must carry equal indices $(a, b; \tau, i)$.

Using these rules it is easy to see that all terms listed in (81) vanish identically except for the term $(4, 2)$. This is because after applying the rules (82) until all δP's and $\delta\sigma$'s have disappeared, we are left with a product of graded traces each involving powers of the propagators $g_\mathrm{d}^{(aa)}$ or $g_\mathrm{d}^{(a(a+1))}$. But each of these two propagators has the form $F_{\mu\nu}\delta_{\alpha\beta} + G_{\mu\nu}L_{\alpha\beta}$. Here, F and G are matrices in Hilbert space, and the indices α and β refer to the graded space. Thus, the graded traces of the g's vanish, and the same statement holds for arbitrary powers of these propagators. For later use, we keep track of the result,

$$\mathrm{trg}[(g_\mathrm{d}^{(aa)})^k] = 0 = \mathrm{trg}[(g_\mathrm{d}^{(a(a+1))})^k] \tag{83}$$

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which applies for all non–negative integers k.

The term $(4, 2)$ can be worked out explicitly. A completely analogous calculation was done in Ref. [11] and is not repeated here. One finds that the term $(4, 2)$ does not vanish for $l \to \infty$ but is of the same order of magnitude as the term $(0, 2)$ worked out in Section 7. This finding is consistent with the general result of Ref. [11] which states that if the interaction has rank k and the number of Fermions is m then the terms in the loop expansion do not vanish if $2k \leq m$. In the present case we have $k = 1$ and $m = 2$. We conclude that our result Eq. (79) for the localization length may be modified by the loop expansion. This is why we now examine terms of arbitrary order in the loop expansion. We do so in order to anwer the question: Does the loop expansion contribute terms which depend on the distance between the box $(a(a + 1))$ and the box $((a + n)(a + n + 1))$, i.e., between the end points of the Green's function in Eq. (9)? If the answer is yes, the behavior of the localization length displayed in Eq. (79) will change; otherwise, the result in Eq. (79) is not affected by the loop expansion.

Prior to performing the Wick contractions, the terms in the loop expansion have the form

$$(p_1, q_1)(p_2, q_2) \times \ldots \times (p_n, q_n) . \tag{84}$$

Here $\sum_i q_i = 2$ while $\sum_i p_i = p \geq 2$ must be even. We have omitted a binomial factor which is due to the Taylor expansions of both, the logarithm and the exponential. We refer to expressions like (84) as to "terms", and to the individual factors (p_i, q_i) as to "brackets". Each bracket with $q_i = 0$ comprises a sum over all boxes. Moreover, each bracket contains a trace over Hilbert space. Therefore, each term in the sum over boxes defines one or several closed loops in the sequence of boxes. For instance, the bracket $(3, 0)$ allows for the realization

$$\sum_a \text{tr trg} \, [\, g_{\mathrm{d}}^{((a-1)a)} \, \sum_\tau \lambda \, C^{(1\tau)}(-1, 0; l) \, \delta\Sigma^{((a-1)a;\tau)} \, g_{\mathrm{d}}^{(aa)}$$

$$\times \sum_{\tau'} \lambda \, C^{(1\tau')}(0, 1; l) \, \delta\Sigma^{(a(a+1);\tau')} \, g_{\mathrm{d}}^{(a(a+1))}$$

$$\times \sum_{\tau''} \lambda \, C^{(1\tau'')}(-1, 1; r) \, [\delta\Sigma^{((a-1)(a+1);\tau'')}]^\dagger \,] . \tag{85}$$

Here and in the sequel, we use the notation introduced in Eqs. (42). In the expression (85), the loop consists of the sequence $((a - 1)a) \to (aa) \to (a(a + 1)) \to ((a - 1)a)$. Another possible realization of the bracket $(3, 0)$ is given by

$$\sum_a \text{tr trg} \, [\, g_{\mathrm{d}}^{(aa)} \delta P^{(aa)} \, g_{\mathrm{d}}^{(aa)} \delta P^{(aa)} \, g_{\mathrm{d}}^{(aa)} \delta P^{(aa)} \,] . \tag{86}$$

Here the loop runs entirely within the box (aa). In general, we speak of a realization of a term if, together with the sequence of brackets (84), also the sequence of δP's and $\delta\sigma$'s is fixed.

The δP's and $\delta\sigma$'s come in two classes. The first class comprises the forms

$$\delta P^{(aa)} \, , \, \delta P^{(a(a+1))} \, , \, \sum_\tau C^{(1\tau)}(0) \, \delta\sigma^{(aa,\tau)} \, , \, \text{and} \, \sum_\tau C^{(1\tau)}(1) \, \delta\sigma^{(a(a+1),\tau)} . \tag{87}$$

These forms connect states within a box to states within the same box. We refer to these forms jointly by the symbol δD where D stands for "diagonal". The second class comprises the forms

$$\sum_\tau C^{(1\tau)}(-1,0;l) \, \delta\Sigma^{((a-1)a,\tau)} \, , \, \sum_\tau C^{(1\tau)}(0,1;l) \, \delta\Sigma^{(a(a+1),\tau)} \, , \text{ and}$$

$$\sum_\tau C^{(1\tau)}(-1,1;l) \, \delta\Sigma^{((a-1)(a+1),\tau)} \tag{88}$$

which connect different boxes and move upward in the sequence of boxes, and the forms

$$\sum_\tau C^{(1\tau)}(-1,0;r) \, [\delta\Sigma^{((a-1)a,\tau)}]^\dagger \, , \, \sum_\tau C^{(1\tau)}(0,1;r) \, [\, \delta\Sigma^{(a(a+1),\tau)} \,]^\dagger \, , \text{ and}$$

$$\sum_\tau C^{(1\tau)}(-1,1;r) \, [\, \delta\Sigma^{((a-1)(a+1),\tau)} \,]^\dagger \tag{89}$$

which move downward in the sequence of boxes. We refer to the first (second) group jointly by the symbol $\delta\Sigma$ ($\delta\Sigma^\dagger$, respectively). Within each realization of a term of the loop expansion, the number of δD's must be even, and to each $\delta\Sigma$ there must correspond a $\delta\Sigma^\dagger$ of the same type.

For each term in the loop expansion, we first study all the realizations which do not carry any $\delta\Sigma$'s and $\delta\Sigma^\dagger$'s. For these realizations, the brackets $(p_i,1)$ vanish because the source term J connects the boxes $(a(a+1))$ and $((a+n)(a+n+1))$ with $n \neq 0$, and there is no way of constructing a closed loop from J and the δD's. We conclude that only terms of the form $(p_1,0) \times \ldots \times (p_{n-1},0)(p_n,2)$ possess the realizations here considered. The brackets $(p_i,0)$ define closed loops within the same box, with a final summation over all boxes. The bracket $(p_n,2)$ consists of a loop with $p_n - k$ steps within the box $(a(a+1))$, followed by the step $(a(a+1)) \to ((a+n)(a+n+1))$ induced by J, followed by k steps within the box $((a+n)(a+n+1))$, followed by the step $((a+n)(a+n+1)) \to (a(a+1))$ induced by the second factor J. Here, $k = 0, 1, \ldots, p_n$. In every bracket $(p_i,0)$ with $i = 1, \ldots, (n-1)$ we split the sum over boxes into two parts. The first part comprises the sum over the boxes $(a(a+1))$ and $((a+n)(a+n+1))$ and is denoted by $(p_i,0)_1$. The sum over all remaining boxes is denoted by $(p_i,0)_2$. The product $(p_1,0) \times \ldots \times (p_{n-1},0)$ is accordingly given by a contribution of the form $(p_1,0)_1 \times \ldots \times (p_{n-1},0)_1$ and another contribution where at least one of the brackets $(p_i,0)$ carries the index 2. The Gaussian average over the second contribution multiplied by $(p_n,2)$ vanishes. Indeed, since each δD is multiplied from the left by a factor g, application of the contraction rules (82) leaves us, after all Wick contractions have been performed, with a product of graded traces of the form (83) and, thus, yields zero. We are left with the contribution $(p_1,0)_1 \times \ldots \times (p_{n-1},0)_1(p_n,2)$. This contribution will, in general, differ from zero and be of the same order in $N_{a(a+1)}$ as the bracket $(0,2)$. The term $(4,0)$ furnishes an example. The chain of terms $(4,2); (4,2)(2,0); (6,2); (6,2)(2,0); (8,2); \ldots$ shows that terms of arbitrary order in p do contribute. Needless to say, we are not able to work out the contributions from all these terms analytically. Performing the

traces in Hilbert space is one of the stumbling blocks. We observe, however, that these contributions have the same form for any value of n. This is because the propagators g, the coefficients $C^{(1\tau)}$ and the weight factors N in the quadratic forms (80) are the same for all boxes $((a + n)(a + n + 1))$. Therefore, the realizations which do not carry any factors $\delta\Sigma$ and $\delta\Sigma^\dagger$ will renormalize the value of the average two–point function but will not affect the dependence of this function on n. In other words, the ratio (79) of localization lengths remains unaffected by such realizations of the loop expansion.

We turn to the realizations which carry at least one factor $\delta\Sigma$ and $\delta\Sigma^\dagger$ each and show that for $n \geq 3$, the contributions from such realizations vanish. For any such realization, we Wick–contract pairs of $\delta\Sigma$'s and $\delta\Sigma^\dagger$'s until only a single such pair is left, leaving the δD's untouched. In general, there are many different ways of performing these Wick contractions. Therefore, every realization gives rise to many different expressions with a single remaining pair $\delta\Sigma, \delta\Sigma^\dagger$ each. Each such expression can be written in terms of the brackets introduced above. Because of the contraction rules (82), the bracket structure is quite different from that of the original term (84). Our claim is proved if we show that every sequence of brackets involving two J's and a single pair $\delta\Sigma, \delta\Sigma^\dagger$ vanishes upon Wick contraction.

The pair $\delta\Sigma, \delta\Sigma^\dagger$ connects two different boxes which we denote by (a, b) and (a', b'). For $n \geq 3$, at least one of these two boxes must differ from both, the box $(a(a + 1))$ and the box $((a + n)(a + n + 1))$. Without loss of generality we assume this to be the box (a, b). For $n \geq 3$ both $\delta\Sigma$ and $\delta\Sigma^\dagger$ must occur within the same bracket. Otherwise, it is not possible to form a closed loop. This bracket contains either the string of factors

$$\ldots \delta\Sigma \sum_\mu |ab\mu\rangle\langle ab\mu|(g_{\rm d}^{(ab)}\delta D^{(ab)})^k g_{\rm d}^{(ab)} \sum_\nu |ab\nu\rangle\langle ab\nu|\delta\Sigma^\dagger \ldots \qquad (90)$$

with k zero or positive integer, or the Hermitean adjoint of the form (90). According to the contraction rules (82), Wick contraction of $\delta\Sigma$ and $\delta\Sigma^\dagger$ yields the factor ${\rm trg}[(g_{\rm d}^{(ab)}\delta D^{(ab)})^k g_{\rm d}^{(ab)}]$. The remaining brackets contain the J's quadratically, an arbitrary number of δD's, but no further $\delta\Sigma$'s or $\delta\Sigma^\dagger$'s. Wick–contracting all the $\delta D^{(ab)}$'s and application of the result (83) yields zero since the box (ab) differs from both $(a(a+1))$ and $((a+n)(a+n+1)$. This establishes our claim.

The difference between the realizations which do not carry any factors $\delta\Sigma$ and $\delta\Sigma^\dagger$ and those which do, lies in the fact that the former possess one variant (the combination $(p_1, 0)_1 \times \ldots \times (p_{n-1}, 0)_1 (p_n, 2)$ where the δD's all carry the box indices $(a(a+1))$ or $((a+n)(a+n+1))$ while for the latter, the occurrence of at least one pair $(\delta\Sigma, \delta\Sigma^\dagger)$ necessarily causes the occurrence of δD's carrying a box index different from both $(a(a+1))$ and $((a+n)(a+n+1))$. With the help of Eq. (83), the Gaussian average over the latter yields zero while in the former case, the last graded trace involves the J's and, therefore, does not vanish.

In conclusion, we have shown that the result (79) for the ratio of localization lengths remains valid to all orders in the loop expansion.

9 Loop Expansion: Terms of Order v^2

At first sight, the result of Section 8 is somewhat puzzling. Indeed, it was pointed out at the end of Section 7 that the result for the ratio of the localization lengths, Eq. (79), cannot be exact. This provided the motivation for the investigation of the loop expansion in Section 8 to zeroth order in v. It turned out, however, that Eq. (79) remains valid without modification to all orders in that loop expansion. This points to the need to investigate other loop corrections (which are definitely needed to yield a physically acceptable result). Such corrections must obviously be of order v^2 and will now be studied. They come in two classes. There are corrections of order v^2 to the source terms, and there are other corrections which are independent of the source terms. It is easy to see that corrections in the first class cannot correct the result Eq. (79) in a physically acceptable way. Indeed, the transport coefficient in Eq. (78) is proportional to $\Delta_0 \, \Delta_1$. This factor must be corrected to become independent of the saddle–point solution. We note that this factor applies to all pairs of boxes $[(aa), (a(a+1)]$ and $[((a-1)a), (aa)]$, for $a = 1, \ldots, K$, and so must, therefore, the correction we are looking for. But by definition of the hopping matrix elements, a term of order v^2 which modifies the source term can affect only the pair of boxes immediately adjacent to the end–point boxes that occur in Eq. (9). In keeping with common practice we do not, therefore, consider corrections in the first class any further.

It turns out that terms in the second class do provide the required corrections. To study such terms, we return to $\mathrm{tr}\,\mathrm{trg}\ln \mathbf{N}(J)$ as defined in Eqs. (32) to (39) with the substitutions (70). To calculate terms of order v^2 which are independent of J, it suffices to consider block–diagonal submatrices $M^{(aa)}$ and $M^{((a-1)a)}$ of $\mathbf{N}(0)$. Here $M^{(aa)}$ contain the boxes (aa) and $(a(a+1))$ in both rows and columns,

$$
\begin{aligned}
\langle aa\mu|M^{(aa)}|aa\nu\rangle &= (E - \lambda\sigma_{\mathrm{sp}}^{(aa)} - \lambda T^{(aa)}\delta P^{(aa)}(T^{(aa)})^{-1})\delta_{\mu\nu} \\
&\quad -w_{\mu\nu} \\
&\quad -\sum_{\tau}\lambda C_{\mu\nu}^{(1\tau)}(0)T^{(aa)}\delta\sigma^{(aa;\tau)}(T^{(aa)})^{-1} \;, \\
\langle aa\mu|M^{(aa)}|a(a+1)\nu\rangle &= -v\delta_{\mu\nu} - (i/2)\sum_{\tau}\lambda C_{\mu\nu}^{(1\tau)}(-1,0;l) \\
&\quad \times T^{(aa)}\delta\Sigma^{(a(a+1);\tau)}(T^{(a(a+1))})^{-1} \;, \\
\langle a(a+1)\mu|M^{(aa)}|aa\nu\rangle &= -v\delta_{\mu\nu} + (i/2)\sum_{\tau}\lambda C_{\mu\nu}^{(1\tau)}(-1,0;r) \\
&\quad \times T^{(aa)}[\delta\Sigma^{(a(a+1);\tau)}]^{\dagger}(T^{(a(a+1))})^{-1} \;, \\
\langle a(a+1)\mu|M^{(aa)}|a(a+1)\nu\rangle &= (E - \lambda\sigma_{\mathrm{sp}}^{(a(a+1))} \\
&\quad -\lambda T^{(a(a+1))}\delta P^{(a(a+1))}(T^{(a(a+1))})^{-1})\delta_{\mu\nu} \\
&\quad -\lambda\sum_{\tau}C_{\mu\nu}^{(1\tau)}(1)T^{(a(a+1))}\delta\sigma^{(a(a+1);\tau)} \\
&\quad\quad \times(T^{(a(a+1))})^{-1} \;,
\end{aligned}
\tag{91}
$$

while $M^{((a-1)a)}$ contains the boxes $((a-1)a)$ and (aa) in both rows and columns and is not displayed explicitly. For the terms of second order in v, we have

$$\text{tr trg} \ln \mathbf{N}(0) \approx \sum_a \text{tr trg } [\ln \ M^{(aa)} + \ln M^{((a-1)a)} \ ] \ . \tag{92}$$

To justify Eq. (92), we observe that contributions which are correct in second order in v can be generated from the matrix $\mathbf{N}(0)$ by dropping in that matrix all elements v but two. The set of matrices generated in this way, when inserted into the expression tr trg ln and expanded into powers of the v's, δP's and $\delta\sigma$'s, produces the same terms of second order in v as tr trg ln $\mathbf{N}(0)$ itself. In those matrices where the two remaining v's do not appear in mirror positions with respect to the main diagonal (as they do in Eqs. (91)), the terms in the power series will not carry all $\delta\Sigma$'s and $[\delta\Sigma]^\dagger$'s in pairs of the same type and will, therefore, vanish upon Wick contraction. Those matrices which do have the two remaining v's appear in mirror positions with respect to the main diagonal will have much larger size than shown in Eqs. (91), containing in addition other boxes connected with the ones shown in Eqs. (91)) by $\delta\Sigma$'s and $[\delta\Sigma]^\dagger$'s. The arguments presented towards the end of Section 8 (see the discussion around expression (90)) apply likewise in the present case, however, and show that contributions from such additional $\delta\Sigma$'s and $[\delta\Sigma]^\dagger$'s vanish. We are left with matrices which are block–diagonal, the diagonal block shown explicitly in Eqs. (91) being disconnected from the remaining one(s). When we evaluate the expression tr trg ln, the two diagonal blocks contribute additively, and only $M^{(aa)}$ produces contributions of second order in v. We see that Eq. (92) holds except for terms which vanish upon Wick contraction and in the sense that the terms of second order generated by expanding either side, agree.

We focus attention on ln $M^{(aa)}$; the terms ln $M^{((a-1)a)}$ are treated analogously. Expanding ln $M^{(aa)}$ in powers of v and the δP's and $\delta\sigma$'s, keeping only the terms of second order in v (but terms of arbitrary order in the δP's and $\delta\sigma$'s), we obtain a series of terms which individually have the form

$$v^2 \text{ tr trg}[A^{(aa)}(T^{(aa)})^{-1}(T^{(a(a+1))})A^{(a(a+1))}(T^{(a(a+1))})^{-1}(T^{(aa)})] \ . \tag{93}$$

The matrix $A^{(aa)}$ has the form (we drop the Hilbert space indices (μ, ν))

$$
\begin{aligned}
A^{(aa)} \ = \ & (g_\text{d}^{(aa)}(\lambda\delta P^{(aa)} + \sum_\tau C^{(1\tau)}(0)\delta\sigma^{(aa;\tau)})^{k_1} g_\text{d}^{(aa)} \\
& \times \sum_\tau \lambda C^{(1\tau)}(-1,0;l)\delta\Sigma^{(a(a+1);\tau)} \\
& \times (g_\text{d}^{(a(a+1))}(\lambda\delta P^{(a(a+1))} + \sum_\tau C^{(1\tau)}(1)\delta\sigma^{(a(a+1);\tau)})^{k_2} g^{(a(a+1))} \\
& \times \sum_\tau \lambda C^{(1\tau)}(-1,0;r)[\delta\Sigma^{(a(a+1);\tau)}]^\dagger \ldots \ , \tag{94}
\end{aligned}
$$

and correspondingly for $A^{(a(a+1))}$. The exponents $k_1, k_2, \ldots$ are positive integer or zero. In both $A^{(aa)}$ and $A^{(a(a+1))}$, $\delta\Sigma$'s and $\delta\Sigma^\dagger$'s with equal indices must always occur in pairs.

We expand the exponent of $\overline{Z}$ in powers of the terms (93), keep only terms of second order in v, perform the Wick contractions, and re–exponentiate the result. This procedure is correct to order v^2. It is easy to see that after Wick contraction, each of the terms (93) either vanishes or is proportional to

$$(v/\lambda)^2 \ \mathrm{trg}(T^{(aa)}L(T^{(aa)})^{-1}T^{(a(a+1))}L(T^{(a(a+1))})^{-1}) \ , \tag{95}$$

and that the dimensionless constant of proportionality does not depend on the box labels (aa) and $(a(a+1))$. We conclude that our procedure does generate the expected renormalization of the transport coefficient $(v/\lambda)^2 \ \Delta_0 \ \Delta_1$ appearing in Eq. (76). It remains to determine the magnitude of the renormalization.

Unfortunately, this cannot be done using the loop expansion. Even the analytical calculation of individual terms beyond the ones of lowest order (let alone that of the entire series) is a prohibitive task. The difficulty lies in performing the traces in Hilbert space. Therefore, we resort to a different approach. We use a perturbative expansion which, in a similar context, was developed in Ref. [17] and which is akin to diagrammatic impurity perturbation theory. This approach cannot give us an explicit expression either for the renormalized transport coefficient. It does allow us, however, to understand the physical origin and meaning of the renormalization terms.

Starting point is the observation that in Eqs. (91), only two boxes play a role. Instead of the original Hamiltonian H in Eq. (3), we, therefore, consider the projection PHP of H onto the part of Hilbert space spanned by the states $|aa\mu\rangle$ and $|a(a+1)\mu\rangle$. We consider the average two–point function

$$\overline{|\langle aa\mu_0|(E^+ - PHP)^{-1}|a(a+1)\nu_0\rangle|^2} \ . \tag{96}$$

To calculate this expression, we first imagine using the technique developed in the present paper. The resulting averaged generating function would be written as an integral over the variables $T^{(aa)}, T^{(a(a+1))}, \delta P^{(aa)}, \delta P^{(a(a+1))}, \delta\sigma^{(aa;\tau)}$, $\delta\sigma^{(a(a+1);\tau)}, \delta\Sigma^{(a(a+1);\tau)}$ and $[\delta\Sigma^{(a(a+1);\tau)}]^\dagger$. Except for a source term which reflects the choice (96) of observable and is irrelevant for what follows, the exponent of the generating function would contain the term $-\mathrm{tr} \ \mathrm{trg} \ \ln M^{(aa)}$ with $M^{(aa)}$ defined in Eq. (91). This fact then connects our renormalization problem to the two–point function defined in (96). Put differently, determining our renormalized transport coefficient is tantamount to determining the transport coefficient connecting the boxes (aa) and $(a(a + 1))$ for the average two–point function (96).

An alternative way of doing the calculation consists in expanding $(E^+ - PHP)^{-1}$ in Eq. (96) in powers of the stochastic variables $h_{ij}^{(a)}$ in Eq. (4). Using the fact that the $h_{ij}^{(a)}$'s are Gaussian distributed random variables with zero mean value, the terms in the resulting series can be averaged using Wick contraction. The resulting contraction patterns come in two classes: (i) those where all contraction lines connect partners within the same box and (ii) those where at least one contraction line connects a partner in box (aa) with a partner in box $(a(a + 1))$. Eq. (16) shows that the second class is not empty. Iteration

of the Pastur equation, Eq. (47), and comparison with our perturbation expansion shows that the saddle–point solution comprises all contraction patterns in class (i) with non–intersecting contraction lines. This is a well–known result. In the terminology of Section 8, the sum of the remaining contraction patterns corresponds to the sum of terms in the loop expansion. Realizations of terms in the loop expansion which are of order zero in $\delta\Sigma$ and $[\delta\Sigma]^\dagger$ correspond to the remaining contraction patterns in class (i). The rest corresponds to class (ii). It was shown in a different context [11] that the remaining contraction patterns in class (i) are needed to change the average level density as given by the saddle–point condition, into the true level density. It is natural to assume that the same statement applies in the present context. Contraction patterns in class (ii) are not encountered frquently. They arise here because the interactions in boxes (aa) and $(a(a+1))$ are correlated, and so must be the level densities $\rho_0(E)$ and $\rho_1(E)$.

As a result of these heuristic arguments, we are led to the conclusion that the loop expansion renormalizes the transport coefficient. The saddle–point version $T_{\rm sp}$ given in Eq. (78) is changed into

$$T = 2\pi \ v^2 \ \langle \rho_0(E)\rho_1(E)\rangle \ . \tag{97}$$

The brackets indicate the average over the product of correlated densities. The ratio (79) of localization lengths changes accordingly into

$$\frac{\zeta(w \neq 0)}{\zeta(w = 0)} = \frac{\langle\rho_0(E)\rho_1(E)\rangle}{\langle(\rho_1(E))^2\rangle} \ . \tag{98}$$

We now take the limit in which the longitudinal thickness d of the slices tends to zero while their number K goes to infinity. This yields

$$\frac{\zeta(w \neq 0)}{\zeta(w = 0)} = \lim_{d\to 0}\left(\frac{\langle\rho_0(E)\rho_1(E)\rangle}{\langle(\rho_1(E))^2\rangle}\right) \ . \tag{99}$$

We expect that the ratio on the right–hand side of Eq. (99) changes very smoothly with d and that a good numerical approximation can be obtained for finite slide thickness.

While we have reached the end of what seems possible by analytical means, the arguments presented above suggest a numerical approach to the problem which is feasible and which can be used to determine both T and the ratio $\zeta(w \neq 0)/\zeta(w = 0)$ for any given two–body interaction w. For each of the two boxes (aa) and $(a(a+1))$, we normalize the level density so that $\int dE\rho(E) = 1$. We use $\rho(E) = (i/(2\pi))(1/N)\mathrm{tr}[G^+(E) - G^-(E)]$ where N stands for the dimension of Hilbert space in either box. Then we can write $\langle\rho_0(E)\rho_1(E)\rangle$ in the form

$$\langle\rho_0(E)\rho_1(E)\rangle = -\frac{1}{4\pi N_{aa}N_{a(a+1)}}$$
$$\times\langle\mathrm{tr}[G^+_{aa}(E) - G^-_{aa}(E)]\mathrm{tr}[G^+_{a(a+1)}(E) - G^-_{a(a+1)}(E)]\rangle \ . \tag{100}$$

30

The w–dependence rests in $[G^+_{aa}(E) - G^-_{aa}(E)]$. The numerical calculation would draw the one–body Hamiltonians for the two particles in the slices a and $(a+1)$ at random, determine the Green's functions $G^+_{aa}(E)$ and $G^+_{a(a+1)}(E)$ by diagonalization of the resulting Hamiltonians in the two boxes, form the expression $\mathrm{tr}[G^+_{aa}(E) - G^-_{aa}(E)]\ \mathrm{tr}[G^+_{a(a+1)}(E) - G^-_{a(a+1)}(E)]$, and repeat the procedure many times so as to generate a meaningful ensemble average. In contrast to a numerical simulation of the full problem involving a large number of boxes, this approach is quite feasible. The simplification which we have achieved by analytical means consists in the restriction of the calculation to two boxes.

10   Discussion and Summary

We have studied the localization properties of two interacting electrons in a disorder potential. Both electrons move within a quasi one–dimensional wire. This wire is thought of as being divided into K slices, with the surfaces separating neighboring slices perpendicular to the axis of the wire. The electrons interact if in the same slice. Hopping matrix elements allow each electron to move from any slice to either of the two neighboring slices. To simplify the problem, we have admitted only those states in Hilbert space where both electrons either occupy the same or two neighboring slices. We have argued that this simplification should be physically irrelevant: The influence of the two–body interaction on localization properties should not depend qualitatively on the omission of states in Hilbert space where the two electrons are two or more slices apart. The limit in which the longitudinal thickness of each slice tends to zero and the number K of slices tends to infinity is considered but turns out not to affect our results in an essential way.

The analytical treatment of this problem becomes possible thanks to the eigenvalue decomposition of the second moment of the matrix elements of the disorder Hamiltonian in Eqs. (25). This is the novel technical feature of our approach. It allows us to use the supersymmetry technique. We calculate the average of the generating function, use the Hubbard-Stratonovich transformation, and determine the saddle–point solutions in more or less standard fashion. The result (78) for the transport coefficient and for the ratio (79) for the localization lengths in the presence and in the absence of the two–body interaction w is intuitively convincing and demonstrates the influence of w.

For the shape of the spectral density of two non–interacting electrons, the saddle–point solution yields the semicircle, a result which is manifestly not exact. This led us to consider loop corrections to the saddle–point solution. Herein lies the second technically novel aspect of our work. We succeeded in exhibiting general properties of all terms in the loop expansion up to arbitrary order. We could show that the loop corrections to the source terms do not alter the results for the transport coefficient and, thus, for the localization length obtained in the framework of the saddle–point approximation. We studied further loop corrections which are independent of the source terms. For these terms we could show to arbitrary order in the loop expansion that each of the contributions which

are of second order in v has precisely the form needed to yield a renormalization correction to the transport coefficient. We were not able to calculate the coefficients multiplying these terms and, thus, the magnitude of the renormalization effect. Instead, we used heuristic arguments to show that the renormalization correction does correspond to physical expectations.

Our main result is embodied in Eq. (99). This is the third novel aspect of our work. We have shown analytically that the two–body interaction does affect the localization length. It does so because it alters the level density for those states in Hilbert space where the two electrons occupy the same slice. The result may be an increase or decrease of the localization length, depending both on properties of the two–body interaction and on the location of the energy E in the spectrum. A sizeable change occurs if the two–body interaction meets the criterion (67). Effectively, this puts an upper bound on the strength of the impurity potential which can easily be checked in practice.

We draw attention to a characteristic difference between the one–electron and the two–electron problem. In the former case, the dimensionless transport coefficient is given in the saddle–point approximation by $2\pi v^2(\Delta_1(E))^2$. The energy E is that of the electron. The result is exact because loop corrections vanish in the limit $N \to \infty$. In the case of two electrons without interaction, the dimensionless transport coefficient has, in the saddle–point approximation, the form $2\pi v^2(\Delta_1(E))^2$. The apparent similarity of both expressions is deceiving because now E is the total energy of the two electrons. Moreover, the result is not exact but modified by contributions from the loop expansion which changes the form of the level density into a convolution of two semicircles. We might, of course, specify the energy of each of the two electrons. But this is not a meaningful thing to do if we wish to compare the localization lengths without and with interaction. In the latter case, the total energy is the only constant of motion.

The supersymmetry method does not apply to one–dimensional problems. Therefore, we cannot compare our result with what has been found numerically in one–dimensional systems. It is possible, however, to compare our work with the result Eq. (1) which is not restricted to one dimension. This expression does not contain the hopping matrix element v. Thus, it differs from our expression (97) in which we retain the structure typical for the Thouless block scaling argument with hopping between boxes but modify the level densities. Moreover, the arguments presented in the previous paragraph show that it is not straightforwardly possible to compare the localization length of the two–electron problem with that of the one–electron problem. We note, however, that we predict a change of localization length which depends upon all moments $\mathrm{tr}(w), \mathrm{tr}(w^2), \ldots$ of the two–body interaction while the result Eq. (1) involves only U^2 and is, thus, independent of the sign of the two–body interaction.

To the best of our knowledge, this is the first time that a complete analytical treatment of the combined effects of disorder and interaction has been possible. We believe to have given a complete analysis of the influence of the two–body interaction on localization properties of two electrons in a quasi one–dimensional disorder potential. This statement is subject to the proviso that we have worked

in a reduced Hilbert space.

11 Appendix 1: Eigenvectors and Eigenvalues of the matrices $\mathcal{A}^{(i)}$

The matrices $\mathcal{A}^{(-1)}$ and $\mathcal{A}^{(-1,0)}$ were treated in Section 3. The remaining matrices are treated correspondingly. Therefore, we simply list eigenvectors and left– and right–hand eigenvalues.

$$
\begin{aligned}
i = (-1): \ & \Lambda^{(0)}(-1) = 2l; \ C^{(0)}_{\mu\nu}(-1;r) = C^{(0)}_{\mu\nu}(-1;l) \propto \delta_{\mu\nu} \ ; \\
& \Lambda^{(1)}(-1) = 2l - 1; \ C^{(1\tau)}_{\mu\nu}(-1;l) = (C^{(1\tau)}_{\mu\nu}(-1;r))^{\dagger} \\
& \propto \langle (a-1)a\mu | \alpha^{\dagger}_{cm} \alpha_{cm'} | (a-1)a\nu \rangle \text{ with } c = (a-1), a \ ; \\
i = (0): \ & \Lambda^{(0)}(0) = 2(l-1); \ C^{(0)}_{\mu\nu}(0;r) = C^{(0)}_{\mu\nu}(0;l) \propto \delta_{\mu\nu} \\
& \Lambda^{(1)}(0) = l - 2; \ C^{(1\tau)}_{\mu\nu}(0;l) = (C^{(1\tau)}_{\mu\nu}(0;r))^{\dagger} \\
& \propto \langle aa\mu | \alpha^{\dagger}_{am} \alpha_{am'} | aa\nu \rangle \ ; \\
i = (+1): \ & \Lambda^{(0)}(+1) = 2l; \ C^{(0)}_{\mu\nu}(+1;r) = C^{(0)}_{\mu\nu}(+1;l) \propto \delta_{\mu\nu} \ ; \\
& \Lambda^{(1)}(+1) = 2l - 1; \ C^{(1\tau)}_{\mu\nu}(+1;l) = (C^{(1\tau)}_{\mu\nu}(+1;r))^{\dagger} \\
& \propto \langle a(a+1)\mu | \alpha^{\dagger}_{cm} \alpha_{cm'} | a(a+1)\nu \rangle \text{ with } c = a, (a+1) \ ; \\
i = (-1,0): \ & \Lambda^{(1)}(-1,0) = l - 1; \\
& C^{(1\tau)}_{\mu\nu}(-1,0;l) \propto \langle aa\mu | \alpha^{\dagger}_{am} \alpha_{(a-1)m'} | (a-1)a\nu \rangle \ ; \\
& C^{(1\tau)}_{\mu\nu}(-1,0;r) \propto \langle (a-1)a\mu | \alpha^{\dagger}_{(a-1)m} \alpha_{am'} | aa\nu \rangle \ ; \\
i = (0,-1): \ & \Lambda^{(1)}(0,-1) = l - 1; \\
& C^{(1\tau)}_{\mu\nu}(0,-1;l) \propto \langle (a-1)a\mu | \alpha^{\dagger}_{(a-1)m} \alpha_{am'} | aa\nu \rangle \ ; \\
& C^{(1\tau)}_{\mu\nu}(0,-1;r) \propto \langle aa\mu | \alpha^{\dagger}_{am} \alpha_{(a-1)m'} | (a-1)a\nu \rangle \ ; \\
i = (0,+1): \ & \Lambda^{(1)}(0,+1) = l - 1; \\
& C^{(1\tau)}_{\mu\nu}(0,+1;l) \propto \langle a(a+1)\mu | \alpha^{\dagger}_{(a+1)m} \alpha_{am'} | aa\nu \rangle \ ; \\
& C^{(1\tau)}_{\mu\nu}(0,+1;r) \propto \langle aa\mu | \alpha^{\dagger}_{am} \alpha_{(a+1)m'} | a(a+1)\nu \rangle \ ; \\
i = (+1,0): \ & \Lambda^{(1)}(+1,0) = l - 1; \\
& C^{(1\tau)}_{\mu\nu}(+1,0;l) \propto \langle aa\mu | \alpha^{\dagger}_{am} \alpha_{(a+1)m'} | a(a+1)\nu \rangle \ ; \\
& C^{(1\tau)}_{\mu\nu}(+1,0;r) \propto \langle a(a+1)\mu | \alpha^{\dagger}_{(a+1)m} \alpha_{am'} | aa\nu \rangle \ ; \\
i = (-1,+1): \ & \Lambda^{(1)}(-1,+1) = l; \\
& C^{(1\tau)}_{\mu\nu}(-1,+1;l) \\
& \quad \propto \langle a(a+1)\mu | \alpha^{\dagger}_{(a+1)m} \alpha_{(a-1)m'} | (a-1)a\nu \rangle \ ; \\
& C^{(1\tau)}_{\mu\nu}(-1,+1;r)
\end{aligned}
$$

$$\propto \langle (a-1)a\mu | \alpha^\dagger_{(a-1)m} \alpha_{(a+1)m'} | a(a+1)\nu \rangle \ ;$$

$$i = (+1, -1): \ \Lambda^{(1)}(+1, -1) = l;$$

$$C^{(1\tau)}_{\mu\nu}(+1, -1; l)$$

$$\propto \langle (a-1)a\mu | \alpha^\dagger_{(a-1)m} \alpha_{(a+1)m'} | a(a+1)\nu \rangle \ ;$$

$$C^{(1\tau)}_{\mu\nu}(+1, -1; r)$$

$$\propto \langle a(a+1)\mu | \alpha^\dagger_{(a+1)m} \alpha_{(a-1)m'} | (a-1)a\nu \rangle \ . \qquad (101)$$

All other eigenvalues are zero. In the first three cases, special attention must be paid to the cases where $m = m'$, see Section 3. For $i = -1, 0 + 1$ the normalization factors of the Kronecker delta's are $l^2, l^2/2, l^2$, respectively. We take $l \gg 1$. Then, we have

$$\Lambda^{(0)}(i) = 2l \text{ for } i = -1, 0, +1;$$
$$\Lambda^{(1)}(i) = l \text{ all } i \text{ except } i = \pm 1 \text{ where } \Lambda^{(1)}(i) = 2l \ . \qquad (102)$$

12 Appendix 2: The second moment of expression (26)

Because of the fact that only neighboring slices are admitted in the Hilbert space defined in Section 2, the expression (26) reduces to

$$-\frac{i}{2} \sum_{a=1}^{K-1} \sum_{\mu\nu} \Psi^*_{a(a+1)\mu} L^{1/2} \langle a(a+1)\mu | H_0 | a(a+1)\nu \rangle L^{1/2} \Psi_{a(a+1)\nu}$$

$$-\frac{i}{2} \sum_{a=1}^{K} \sum_{\mu\nu} \Psi^*_{aa\mu} L^{1/2} \langle aa\mu | H_0 | aa\nu \rangle L^{1/2} \Psi_{aa\nu} \ . \qquad (103)$$

After averaging over the ensemble $\{H_0\}$, the generating functional contains in the exponent a term given by $1/2$ times the mean value of the square of the expression (103). We use the notation introduced in Section 3 and in Appendix 1 and obtain for this term the following contributions. There are two "diagonal" terms given by

$$-\frac{\lambda^2}{8l} \sum_{a=1}^{K-1} \sum_{\mu\nu\rho\sigma} \Psi^*_{a(a+1)\mu} L \Psi_{a(a+1)\sigma} \Psi^*_{a(a+1)\rho} L \Psi_{a(a+1)\nu} \mathcal{A}^{(1)}_{\mu\nu\rho\sigma} \ ,$$

$$-\frac{\lambda^2}{8l} \sum_{a=1}^{K} \sum_{\mu\nu\rho\sigma} \Psi^*_{aa\mu} L \Psi_{aa\sigma} \Psi^*_{aa\rho} L \Psi_{aa\nu} \mathcal{A}^{(0)}_{\mu\nu\rho\sigma} \ . \qquad (104)$$

A diagonal term containing $\mathcal{A}^{(-1)}$ does not appear and would be redundant. There are six "non–diagonal" terms given by

$$-\frac{\lambda^2}{8l} \sum_{a=2}^{K} \sum_{\mu\nu\rho\sigma} \Psi^*_{aa\mu} L \Psi_{aa\sigma} \Psi^*_{a(a-1)\rho} L \Psi_{a(a-1)\nu} \mathcal{A}^{(0,-1)}_{\mu\nu\rho\sigma} \ ,$$

$$-\frac{\lambda^2}{8l} \sum_{a=2}^{K} \sum_{\mu\nu\rho\sigma} \Psi^*_{(a-1)a\mu} L \Psi_{(a-1)a\sigma} \Psi^*_{aa\rho} L \Psi_{aa\nu} \mathcal{A}^{(-1,0)}_{\mu\nu\rho\sigma} \ ,$$

$$-\frac{\lambda^2}{8l} \sum_{a=2}^{K-1} \sum_{\mu\nu\rho\sigma} \Psi^*_{(a-1)a\mu} L \Psi_{(a-1)a\sigma} \Psi^*_{a(a+1)\rho} L \Psi_{a(a+1)\nu} \mathcal{A}^{(-1,1)}_{\mu\nu\rho\sigma} \ ,$$

$$-\frac{\lambda^2}{8l} \sum_{a=2}^{K-1} \sum_{\mu\nu\rho\sigma} \Psi^*_{a(a+1)\mu} L \Psi_{a(a+1)\sigma} \Psi^*_{(a-1)a\rho} L \Psi_{(a-1)a\nu} \mathcal{A}^{(1,-1)}_{\mu\nu\rho\sigma} \ ,$$

$$-\frac{\lambda^2}{8l} \sum_{a=1}^{K-1} \sum_{\mu\nu\rho \ sigma} \Psi^*_{aa\mu} L \Psi_{aa\sigma} \Psi^*_{a(a+1)\rho} L \Psi_{a(a+1)\nu} \mathcal{A}^{(0,1)}_{\mu\nu\rho\sigma} \ ,$$

$$-\frac{\lambda^2}{8l} \sum_{a=1}^{K-1} \sum_{\mu\nu\rho\sigma} \Psi^*_{a(a+1)\mu} L \Psi_{a(a+1)\sigma} \Psi^*_{aa\rho} L \Psi_{aa\nu} \mathcal{A}^{(1,0)}_{\mu\nu\rho\sigma} \ . \tag{105}$$

For the sum of the terms in expressions (104) and (105), we write somewhat symbolically

$$-\frac{\lambda^2}{8l} \sum_{a} \sum_{i\neq(-1)} \sum_{\mu\nu\rho\sigma} \sum_{\alpha\beta} \Psi^*_{\mu\alpha} L_{\alpha\alpha} \Psi_{\sigma\alpha} \Psi^*_{\rho\beta} L_{\beta\beta} \Psi_{\nu\beta} \mathcal{A}^{(i)}_{\mu\nu\rho\sigma} \ . \tag{106}$$

We have omitted the label indices $a, (a \pm 1)$ on the Ψ's. These, however, are implied by the factors $\mathcal{A}^{(i)}$. We have likewise omitted the limits on the summation over a. We have instead indicated the summation over the graded indices α and β. We now use the decomposition in Eq. (25) for the $\mathcal{A}^{(i)}$'s and obtain several contributions. The first one is due to the terms with $s = 0$ and given by

$$\frac{1}{2l^2} \sum_{a=1}^{K} \mathrm{trg}[(A^{(aa)}(0))^2] + \frac{1}{4l^2} \sum_{a=1}^{K-1} \mathrm{trg}[(A^{(a(a+1))}(1))^2] \ . \tag{107}$$

The graded matrices $A^{(aa)}_{\alpha\beta}$ and $A^{(a(a+1))}_{\alpha\beta}$ are defined by

$$A^{(aa)}_{\alpha\beta} = i\lambda \sum_{\mu} L^{1/2}_{\alpha\alpha} \Psi_{aa\mu\alpha} \Psi^*_{aa\mu\beta} L^{1/2}_{\beta\beta} \ ,$$

$$A^{(a(a+1))}_{\alpha\beta} = i\lambda \sum_{\mu} L^{1/2}_{\alpha\alpha} \Psi_{a(a+1)\mu\alpha} \Psi^*_{a(a+1)\mu\beta} L^{1/2}_{\beta\beta} \ . \tag{108}$$

The second one is due to the terms with $s = 1$ and $i = 0, i = 1$. We find

$$\frac{1}{4l^2} \sum_{a=1}^{K} \sum_{\tau} \mathrm{trg}[(A^{(aa;\tau)})^2] + \frac{1}{4l^2} \sum_{a=1}^{K-1} \sum_{\tau} \mathrm{trg}[(A^{(a(a+1);\tau)})^2] \ . \tag{109}$$

The graded matrices $A^{(aa;\tau)}_{\alpha\beta}(0)$ and $A^{(a(a+1);\tau)}_{\alpha\beta}(1)$ are defined by

$$A^{(aa;\tau)}_{\alpha\beta} = i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha} \Psi_{aa\nu\alpha} \Psi^*_{aa\mu\beta} L^{1/2}_{\beta\beta} C^{(1\tau)}_{\mu\nu}(0) \ ,$$

$$A^{(a(a+1);\tau)}_{\alpha\beta} = i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha} \Psi_{a(a+1)\nu\alpha} \Psi^*_{a(a+1)\mu\beta} L^{1/2}_{\beta\beta} C^{(1\tau)}_{\mu\nu}(1) \ . \tag{110}$$

The terms with $s = 1$, $i = (-1, 0)$ and $i = (0, -1)$ are given by

$$\frac{1}{8} \sum_{a=2}^{K} N(-1, 0)^{-1} \sum_{\tau} \text{trg}[A^{((a-1)a;\tau)}(l) A^{((a-1)a;\tau)}(r)$$
$$+ A^{((a-1)a,\tau)}(l) A^{((a-1)a;\tau)}(r)] . \qquad (111)$$

We have used that $N(-1, 0) = N(0, -1)$. The graded matrices are defined by

$$A_{\alpha\beta}^{((a-1)a;\tau)}(-1, 0; l) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{aa\nu\alpha} \Psi_{(a-1)a\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(-1, 0; l) ,$$

$$A_{\alpha\beta}^{((a-1)a;\tau)}(-1, 0; r) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{(a-1)a\nu\alpha} \Psi_{aa\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(-1, 0; r) ,$$

$$A_{\alpha\beta}^{((a-1)a;\tau)}(0, -1; l) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{(a-1)a\nu\alpha} \Psi_{aa\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(0, -1; l) ,$$

$$A_{\alpha\beta}^{((a-1)a;\tau)}(0, -1; r) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{aa\nu\alpha} \Psi_{(a-1)a\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(0, -1; r) .$$

$$(112)$$

The terms with $s = 1$ and $i = (0, 1)$ and $i = (1, 0)$ yield correspondingly

$$\frac{1}{8} \sum_{a=1}^{K-1} N(1, 0)^{-1} \sum_{\tau} \text{trg}[A^{(a(a+1);\tau)}(0, 1; l) A^{(a(a+1);\tau)}(0, 1; r)$$
$$+ A^{(a(a+1),\tau)}(1, 0; l) A^{((a-1)a;\tau)}(1, 0; r)] , \qquad (113)$$

with

$$A_{\alpha\beta}^{(a(a+1);\tau)}(0, 1; l) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{a(a+1)\nu\alpha} \Psi_{aa\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(0, 1; l) ,$$

$$A_{\alpha\beta}^{(a(a+1);\tau)}(0, 1; r) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{aa\nu\alpha} \Psi_{a(a+1)\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(0, 1; r) ,$$

$$A_{\alpha\beta}^{(a(a+1);\tau)}(1, 0; l) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{aa\nu\alpha} \Psi_{a(a+1)\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(1, 0; l) ,$$

$$A_{\alpha\beta}^{(a(a+1);\tau)}(1, 0; r) = i\lambda \sum_{\mu\nu} L_{\alpha\alpha}^{1/2} \Psi_{a(a+1)\nu\alpha} \Psi_{aa\mu\beta}^{*} L_{\beta\beta}^{1/2} C_{\mu\nu}^{1\tau}(1, 0; r) .$$

$$(114)$$

Finally, the terms with $s = 1$ and $i = (-1, 1)$ and $i = (1, -1)$ yield

$$\frac{1}{8} \sum_{a=2}^{K-1} N(-1, 1)^{-1} \sum_{\tau} \text{trg}[A^{((a-1)(a+1);\tau)}(-1, 1; l) A^{((a-1)(a+1);\tau)}(-1, 1; r)$$
$$+ A^{((a-1)(a+1),\tau)}(1, -1; l) A^{((a-1)(a+1);\tau)}(1, -1; r)] , \qquad (115)$$

with

$$A^{((a-1)(a+1);\tau)}_{\alpha\beta}(-1,1;l) = i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha}\Psi_{a(a+1)\nu\alpha}\Psi^*_{(a-1)a\mu\beta}L^{1/2}_{\beta\beta}C^{1\tau}_{\mu\nu}(-1,1;l) ,$$

$$A^{((a-1)(a+1);\tau)}_{\alpha\beta}(-1,1;r) = i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha}\Psi_{(a-1)a\nu\alpha}\Psi^*_{a(a+1)\mu\beta}L^{1/2}_{\beta\beta}C^{1\tau}_{\mu\nu}(-1,1;r) ,$$

$$A^{((a-1)(a+1);\tau)}_{\alpha\beta}(1,-1;l) = i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha}\Psi_{(a-1)a\nu\alpha}\Psi^*_{a(a+1)\mu\beta}L^{1/2}_{\beta\beta}C^{1\tau}_{\mu\nu}(1,-1;l) ,$$

$$A^{((a-1)(a+1);\tau)}_{\alpha\beta}(1,-1;r) = i\lambda \sum_{\mu\nu} L^{1/2}_{\alpha\alpha}\Psi_{a(a+1)\nu\alpha}\Psi^*_{(a-1)a\mu\beta}L^{1/2}_{\beta\beta}C^{1\tau}_{\mu\nu}(1,-1;r) .$$

$$(116)$$

Eqs. (101) show that $C^{(1\tau)}_{\mu\nu}(-1,0;l) = C^{(1\tau)}_{\mu\nu}(0,-1;r)$ and correspondingly for the pairs $i = (0,1), i = (1,0)$ and $i = (-1,1), i = (1,-1)$. The definitions in Eqs. (112,114) and (116) imply that $A^{((a-1)a;\tau)}(-1,0;l) = A^{((a-1)a;\tau)}(1,0;r)$ and similar relations for the remaining five pairs of A's. Therefore, the sum of the terms in expressions (111,113,115) simplifies to

$$\frac{1}{4}\sum_{a=2}^{K} N(-1,0)^{-1} \sum_{\tau} \text{trg}[A^{((a-1)a;\tau)}(-1,0;l)A^{((a-1)a;\tau)}(-1,0;r)$$

$$+\frac{1}{4}\sum_{a=2}^{K-1} N(-1,1)^{-1} \sum_{\tau} \text{trg}[A^{((a-1)(a+1);\tau)}(-1,1;l)A^{((a-1)(a+1);\tau)}(-1,1;r)$$

$$+\frac{1}{4}\sum_{a=1}^{K-1} N(1,0)^{-1} \sum_{\tau} \text{trg}[A^{(a(a+1);\tau)}(0,1;l)A^{(a(a+1);\tau)}(0,1;r) . \qquad (117)$$

The sum of the terms in expressions (107,109) and (117) is equal to 1/2 times the second moment of expression (26). This is the result given in Eq. (27).

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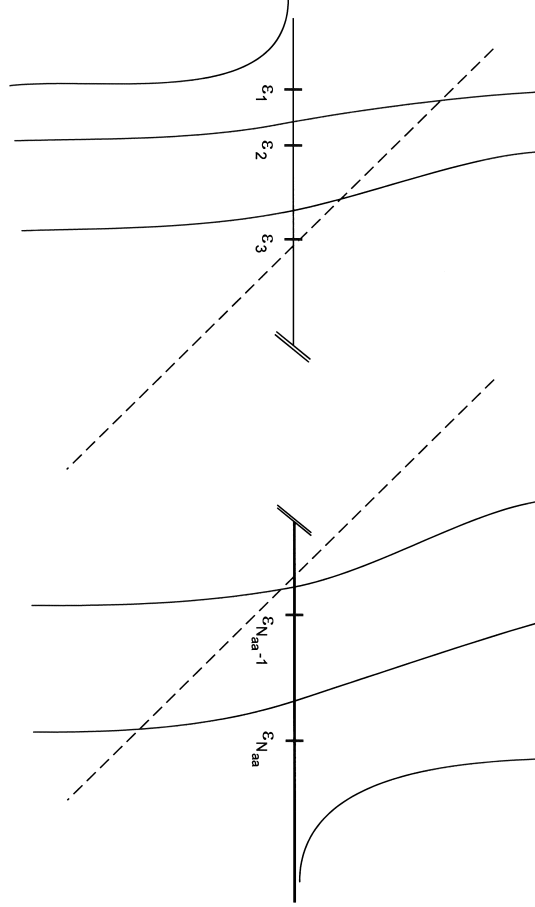[18] C. Itzykson and J.-B. Zuber, Quantum Field Theory, McGraw–Hill, New York 1980

Figure 1: Graphical solution (schematic) of the saddle–point equation, Eq. (64). Some of the eigenvalues ε_j are shown on the abscissa. The τ–dependence of the right–hand side of Eq. (64) is indicated by the solid lines. The two dashed straight lines represent the left–hand side of Eq. (64) for two values of z.